Name:

Instructions:

- This will be a 55-minute exam with 3 problems. Closed-book. No notes/homework.
- Provide your solutions in a simple and clear manner. In particular, you should be able to read them out aloud without any modification.

1. Let $\{X_n\}_{n\in\mathbb{N}}$ be independent random variables such that

$$\mathbb{P}(X_n = -2^n) = 2^{-n} \text{ and } \mathbb{P}(X_n = 2^{-n}) = \mathbb{P}(X_n = 0) = \frac{1}{2}\left(1 - 2^{-n}\right).$$

   Does $S_n := X_1 + \cdots + X_n$ converge almost surely to a finite random variable? Justify your answers.

   Hint: you may use the fact that for a sequence of non-negative real numbers $\{a_n\}_{n\in\mathbb{N}}$, $\lim_{n\to\infty}\sum_{i=1}^n a_i$ exists as either a finite positive constant or infinite.

2. Let $X_1, X_2, \dots$ be i.i.d. random variables with $\mathbb{P}(X_1 = 0) = \mathbb{P}(X_1 = 1) = 1/2$. Consider $Y_n := X_n + X_{2n}$ and $T_n := Y_1 + \cdots + Y_n, n \in \mathbb{N}$.

   Find a sequence of real numbers $\{a_n\}_{n\in\mathbb{N}}$ and prove

$$\frac{T_n}{a_n} \to 1 \text{ in probability.}$$

3. Let $X_1, X_2, \dots$ be i.i.d. standard normal random variables. That is,

$$\mathbb{P}(X_1 \le x) = \int_{-\infty}^{x} \phi(y)dy \quad \text{with} \quad \phi(y) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right).$$

   It is known that

$$\frac{x}{1+x^2}\phi(x) < \mathbb{P}(X_1 > x) < \frac{1}{x}\phi(x) \text{ for all } x > 0.$$

   Show that

$$\limsup_{n\to\infty} \frac{X_n}{\sqrt{2\log n}} = 1 \text{ a.s.}$$

4. Let $\{Z_n\}_{n\in\mathbb{N}}$ be independent random variables, such that for each $n \in \mathbb{N}$, Z_n is a Gaussian random variable with zero mean and variance n^2. Consider $X_n := \exp Z_n$, $n \in \mathbb{N}$. The so-defined X_n is said to have *log-normal distribution.* Consider

$$W_n := X_1 X_2^{1/2} \cdots X_n^{1/n}.$$

   Find the limit of $(\log W_n)/n$.

Solution to problem 3. It suffices to show, for all $\epsilon > 0$,

$$\mathbb{P}\left(\frac{X_n}{\sqrt{2\log n}} > 1+\epsilon \text{ i.o.}\right) = 0 \tag{1}$$

and

$$\mathbb{P}\left(\frac{X_n}{\sqrt{2\log n}} > 1-\epsilon \text{ i.o.}\right) = 1. \tag{2}$$

We first prove (1). By the inequality given,

$$\begin{aligned}\mathbb{P}\left(\frac{X_n}{\sqrt{2\log n}} > 1+\epsilon\right) &= \mathbb{P}\left(X_n > (1+\epsilon)\sqrt{2\log n}\right)\\ &\le \frac{1}{(1+\epsilon)\sqrt{2\log n}}\frac{1}{\sqrt{2\pi}}\exp\left(-(1+\epsilon)^2\log n\right) < \frac{1}{n^{(1+\epsilon)^2}}\end{aligned}$$

for $n \ge 2$. The upper bound is therefore summable over n. That is,

$$\sum_{n=1}^{\infty}\mathbb{P}\left(\frac{X_n}{\sqrt{2\log n}} > 1+\epsilon\right) < 1 + \sum_{n=2}^{\infty}\frac{1}{n^{(1+\epsilon)^2}} < \infty.$$

The first Borel–Cantelli lemma implies (1).

Now to prove (2), similarly, we have

$$\begin{aligned}\mathbb{P}\left(\frac{X_n}{\sqrt{2\log n}} > 1-\epsilon\right) &= \mathbb{P}\left(X_n > (1-\epsilon)\sqrt{2\log n}\right)\\ &> \frac{(1-\epsilon)\sqrt{2\log n}}{1+(1-\epsilon)^2 2\log n}\exp\left(-(1-\epsilon)^2\log n\right) > \frac{(1-\epsilon)}{4\log n}\frac{1}{n^{(1-\epsilon)^2}}\end{aligned}$$

for all $n \ge 2$, $\epsilon > 0$. Therefore,

$$\sum_{n=1}^{\infty}\mathbb{P}\left(\frac{X_n}{\sqrt{2\log n}} > 1-\epsilon\right) > \frac{1-\epsilon}{4}\sum_{n=2}^{\infty}\frac{1}{n^{(1-\epsilon)^2}\log n} = \infty.$$

Since $\{X_n\}_{n\in\mathbb{N}}$ are independent, (2) now follows from the second Borel–Cantelli lemma. □