Name:

Instructions:

- This is a 55-minute exam. Closed-book. No notes/homework.
- Provide your solutions in a simple and clear manner. In particular, you should be able to read them out aloud without any modification.

1. Consider the probability space $([0,1], \mathcal{B}([0,1]), \mathbb{P} \equiv \text{Leb})$. Construct a random variable X on this space with Poisson distribution with parameter λ, that is,
$$\mathbb{P}(X = k) = \frac{\lambda^k}{k!}e^{-\lambda}, k = 0, 1, \ldots.$$
Hint: consider X in the form of $X(\omega) = \sum_{n=1}^\infty a_n \mathbf{1}_{[b_n, c_n)}(\omega)$.

2. Let $\{X_n\}_{n\in\mathbb{N}}$ be a collection of random variables. Suppose there exists a constant $M < \infty$ such that $X_n \le M$ with probability one for all $n \in \mathbb{N}$. Show that
$$\limsup_{n\to\infty} \mathbb{E}X_n \le \mathbb{E}\left(\limsup_{n\to\infty} X_n\right).$$

3. Let X be a random variable with $\mathbb{E}X = 0, \mathrm{Var}X = \sigma^2 < \infty$. Show that
$$\mathbb{P}(X \ge a) \le \frac{\sigma^2}{\sigma^2 + a^2}, a > 0.$$
Hint: Apply Chebychev's inequality with $\varphi(x) = (x + c)^2$ for some $c > 0$. Then you shall arrive at an inequality in the form of
$$\mathbb{P}(X \ge a) \le b_{a,\sigma^2}(c),$$
with an explicit upper bound $b_{a,\sigma^2}(c)$ depending on a, σ^2, c. Minimize your expression $b_{a,\sigma^2}(c)$ in c to get the desired bound.

4. Let X be a non-negative discrete random variable, taking values in $\mathbb{N}$. Show that
$$\mathbb{E}X = \sum_{n=1}^\infty \mathbb{P}(X \ge n).$$