Name:

Instructions:

- This is a two-hour exam. Closed-book. No notes/homework.

1. Let X and $\{X_n\}_{n\in\mathbb{N}}$ be non-negative random variables. Suppose $X_n \downarrow X$ as $n \to \infty$ almost surely. Does it necessarily follow that

$$\lim_{n\to\infty} \mathbb{E}X_n = \mathbb{E}X?$$

Prove or disprove by providing a counterexample.

2. Let $X_1, X_2, \ldots$ be i.i.d. standard normal random variables. That is,

$$\mathbb{P}(X_1 \le x) = \int_{-\infty}^{x} \phi(y)dy \text{ with } \phi(y) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right).$$

It is known that

$$\frac{x}{1+x^2}\phi(x) < \mathbb{P}(X_1 > x) < \frac{1}{x}\phi(x) \text{ for all } x > 0.$$

Show that for all $\epsilon > 0$,

$$\mathbb{P}\left(\frac{X_n}{\sqrt{2\log n}} > 1 - \epsilon \text{ i.o.}\right) = 1.$$

3. Suppose the price of a stock changes by $(1+V_n)\cdot 100\%$ every year, with $\{V_n\}_{n\in\mathbb{N}}$ i.i.d. taking values uniformly from $(-0.05, 0.05)$. Consider the geometric average of the price changes after n years

$$W_n = \left(\prod_{i=1}^{n}(1+V_i)\right)^{1/n}.$$

(a) Show that $\lim_{n\to\infty} \log W_n = c$ almost surely.

(b) Mr. Jackson decides, if $c > 0$, then the stock is worth investigating. Is it?

Hint: you may want to use Jensen inequality.

4. Let $\{X_n\}_{n\in\mathbb{N}}$ be i.i.d. random variables with $\mathbb{P}(X_n = \pm 1) = 1/2, n \in \mathbb{N}$. Find the values of β and $\sigma^2 > 0$ such that

$$\frac{1}{n^\beta}\sum_{k=1}^{n} kX_k \Rightarrow \mathcal{N}(0, \sigma^2).$$

Justify your answer.