Due Mon March 28 in class.
Problems with (∗) are required but no need to hand in.

1. (∗) Read lecture notes 2.3 – 2.4, or corresponding materials from textbook.

2. (∗) Let f certain measurable function with $\int_0^1 |f(x)|dx < \infty$. Let $\{U_n\}_{n\in\mathbb{N}}$ be i.i.d. uniform $(0,1)$ random variables. Show that

$$\lim_{n\to\infty} \frac{f(U_1)+\cdots+f(U_n)}{n} = \int_0^1 f(x)dx, \text{ a.s.}$$

Comment: in practice, most statistics softwares can generate i.i.d. uniform random variables. The result proved here can be used to compute the value of $\int_0^1 f(x)dx$ numerically.

3. Let $X_1, X_2, \ldots$ be i.i.d. random variables, $m \in \mathbb{N}$ and $f : \mathbb{R}^m \to \mathbb{R}$ a bounded measurable function. Consider

$$Y_n := f(X_n, \ldots, X_{n+m-1}).$$

Consider $T_n := Y_1 + \cdots + Y_n$.

(a) Random variables $\{Y_n\}_{n\in\mathbb{N}}$ have the following property: there exists some $\ell \in \mathbb{N}$, such that for all $i \neq j, |i-j| > \ell$, Y_i and Y_j are independent. In such a case, $\{Y_n\}_{n\in\mathbb{N}}$ are referred to as *ℓ-dependent.* What is the smallest value ℓ in this example?

(b) Compute $\mathrm{Var}(T_n)$ and use Markov inequality to show

$$\lim_{n\to\infty} \frac{T_n}{n} = \mu \text{ in probability}$$

for some $\mu \in \mathbb{R}$. Express μ.
Hint: First show $\mathrm{Cov}(Y_i, Y_j) = \varphi(i-j)$ for some function φ. For what values of i do we have $\varphi(i) = 0$?

(c) Prove that

$$\lim_{n\to\infty} \frac{T_n}{n} = \mu \text{ a.s.}$$

Hint: consider $T_n^{(\ell)} = Y_\ell + Y_{m+\ell} + \cdots + Y_{(n-1)m+\ell}, \ell = 1, \ldots, m$. What can you say about $T_n^{(\ell)}/n$ as $n \to \infty$?

Comments: here, part (c) is a strictly stronger result than part (b). However, the method in part (c) depends crucially on the ℓ-dependence assumption, while the method in part (b) is more general.

4. Let $X_1, X_2, \ldots$ be i.i.d. random variables with exponential distribution: $\mathbb{P}(X_1 > x) = e^{-x}, x > 0$. We prove

$$\limsup_{n\to\infty} \frac{X_n}{\log n} = 1 \text{ a.s.}$$

For this purpose, we proceed in two steps.

(a) Show
$$\mathbb{P}\left(\frac{X_n}{\log n} > 1+\epsilon \text{ i.o.}\right) = 0, \text{ for all } \epsilon > 0.$$

(b) Show
$$\mathbb{P}\left(\frac{X_n}{\log n} \geq 1-\epsilon \text{ i.o.}\right) = 1, \text{ for all } \epsilon > 0.$$

5. Suppose a light bulb in the math department lounge burns for an amount of time X, and then remains burned out for an amount of time Y until being replaced. Let X_i and Y_i denote the corresponding times for the i-th light bulb. All these random variables are assumed to be independent. Let R_t denote the amount of time during the period $[0, t]$ such that the light bulb is working. Show that
$$\lim_{t\to\infty} \frac{R_t}{t} = \frac{\mathbb{E}X_1}{\mathbb{E}X_1 + \mathbb{E}Y_1} \text{ a.s.}$$

   (a) Consider $Z_n := X_n + Y_n, n \in \mathbb{N}, S_n := Z_1 + \cdots + Z_n, S_0 := 0$, and
   $$N_t := \sup\{n \in \mathbb{N} \cup \{0\} : S_n \leq t\}, t > 0.$$
   Consider $T_n := X_1 + \cdots + X_n, T_0 := 0$. Then one can write
   $$R_t = T_{N_t} + Y_t \tag{1}$$
   for some non-negative random variable Y_t. Express Y_t in terms of random variables X, Y, S, T and N.

   (b) From (1) it follows that $T_{N_t} \leq R_t < T_{N_t+1}$ almost surely. Prove the desired result. Hint: write
   $$\frac{T_{N_t}}{t} = \frac{T_{N_t}}{N_t} \cdot \frac{N_t}{t}.$$

6. Mr. Smith decided to investigate a total wealth of $W_0 = w > 0$ (in dollars) from next year. By the end of n-th year, his investment becomes W_n, and he reinvestigates all W_n at the beginning of the next year, using the same strategy. His strategy at the beginning of each year is the following: (a) a total $p \cdot 100\%$ of his wealth is spent to buy bonds, which yields \$$a$ for each \$1 investigated by the end of the year; (b) the rest $(1-p) \cdot 100\%$ is spent to buy stocks, which yields V_n for each \$1 investigated by the end of the year. In short, we have
$$W_n := (ap + (1-p)V_n)W_{n-1}, n \in \mathbb{N}.$$
Assume $a > 0$, $p \in (0, 1)$ and $\{V_n\}_{n\in\mathbb{N}}$ are i.i.d. non-negative random variables.

   (a) Show that $\lim_{n\to\infty} n^{-1} \log W_n = c$ almost surely for some constant c. Provide an expression of c.

   (b) Suppose $\mathbb{P}(V_1 = 1) = \mathbb{P}(V_1 = 4) = 1/2$. The c depends only on a and p. Determine the optimal investment strategy p as a function of a, so that c is maximized.