Due Wed March 14 in class.
Problems with $(*)$ are required but no need to hand in.

1. $(*)$ Read lecture notes 2.1 – 2.2, or corresponding materials from textbook.

2. Let X and Y be two non-negative random variables defined on a common probability space. Show that

$$\mathbb{E}\left(\frac{1}{XY}\right) = \int_0^\infty \int_0^\infty \frac{\mathbb{P}(X \le x, Y \le y)}{x^2 y^2} dx dy.$$

   Hint: Fubini's theorem.

3. Recall the definition of convergence in probability: we say X_n converges to X, if

$$\lim_{n\to\infty} \mathbb{P}(|X_n - X| > \epsilon) = 0, \text{ for all } \epsilon > 0. \tag{1}$$

   Consider another condition:

$$\lim_{n\to\infty} \mathbb{P}(X_n \ne X) = 0. \tag{2}$$

   Show that (2) implies (1), but the opposite is not necessarily true without further assumption.

4. Let $X_1, X_2, \ldots$ be random variables and a function $\varphi : \mathbb{Z} \to \mathbb{R}$ such that $\operatorname{Cov}(X_i, X_j) = \varphi(i - j)$. Write $S_n = X_1 + \cdots + X_n$. Show that

$$\operatorname{Var}(S_n) = \sum_{i=-n+1}^{n-1} (n - |i|)\varphi(i) = n\varphi(0) + 2\sum_{i=1}^{n-1}(n-i)\varphi(i).$$

5. Let $X_1, X_2, \ldots$ be uncorrelated random variables such that $\mathbb{E}X_i = \mu_i$ and $\lim_{i\to\infty} \operatorname{Var}(X_i)/i = 0$. Consider $S_n = X_1 + \cdots + X_n$, and $\nu_n = \mathbb{E}S_n/n$. Show that $S_n/n - \nu_n \to 0$ in L^2 and in probability.

   Hint: Compute $\operatorname{Var}(S_n)$ and find an upper bound for $\operatorname{Var}(S_n)/n^2$. You may use the fact that for any sequence of real numbers $\{a_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} a_n = a \in \mathbb{R}$,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n a_i = a.$$

6. Let $X_1, X_2, \ldots$ be i.i.d. random variables with distribution determined by $\mathbb{P}(X_1 = (-1)^k k) = C^{-1}(k^2 \log k)^{-1}, k \ge 2$, with $C = \sum_{k=2}^\infty (k^2 \log k)^{-1}$.

   (a) Show $\mathbb{E}|X_1| = \infty$.

   (b) Find a sequence μ_n such that $S_n/n - \mu_n \to 0$ in probability.
   Hint: Theorem 2.2.7 from textbook.

(c) Find a value μ such that $\lim_{n\to\infty} \mu_n = \mu$ and conclude that

$$\lim_{n\to\infty} S_n/n = \mu \text{ in probability.}$$

You may use the fact $\lim_{n\to\infty} \sum_{k=1}^{n} (-1)^k (k \log k)^{-1}$ exists and is finite.

7. Let $\{A_n\}_{n\in\mathbb{N}}$ be a collection of events, and consider the sets $\{A_n \text{ i.o.}\}$, $\{A_n^c \text{ i.o.}\}$, $\{A_n \text{ eventually}\}$ and $\{A_n^c \text{ eventually}\}$. Which pair of sets are complements of each other? Find all the pairs.

8. (*) Read the proofs of two Borel–Cantelli lemmas.

9. Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of Bernoulli random variables with parameter $\{p_n\}_{n\in\mathbb{N}}$, respectively.

   (a) Show that $X_n \to 0$ in probability, if and only if $\lim_{n\to\infty} p_n = 0$. Hint: for the 'only if' part, prove by contradiction: if not, then there exists an increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ and certain number $\delta > 0$, such that $p_{n_k} > \delta$ for all $k \in \mathbb{N}$. Derive a contradiction from here.

   (b) Show that $X_n \to 0$ almost surely, if $\sum_{n=1}^{\infty} p_n < \infty$.

   (c) Show that if $\{X_n\}_{n\in\mathbb{N}}$ are in addition independent, then $X_n \to 0$ almost surely only if $\sum_{n=1}^{\infty} p_n < \infty$.

   (d) Given any sequence $\{p_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} p_n = 0$, provide a concrete example of $\{X_n\}_{n\in\mathbb{N}}$ defined on an appropriate probability space, such that $X_n \to 0$ almost surely.