Due Mon Feb 22 in class.
Problems with $(*)$ are required but no need to hand in.

1. $(*)$ Read lecture notes 1.6 – 1.7, or corresponding materials from textbook.

2. Let $\{a_{m,n}\}_{m,n\in\mathbb{N}}, \{\alpha_n\}_{n\in\mathbb{N}}$ be two collections of real numbers such that
$$\lim_{n\to\infty} a_{m,n} = \alpha_m, m \in \mathbb{N} \quad \text{and} \quad \sup_{n\in\mathbb{N}} \sum_{m=1}^{\infty} |a_{m,n}| < \infty.$$
Does the following statement
$$\lim_{n\to\infty} \sum_{m=1}^{\infty} a_{m,n} = \sum_{m=1}^{\infty} \alpha_m$$
necessarily follow? Justify your answer.

3. Find an example of functions $\{f_n\}_{n\in\mathbb{N}}$ on $((0,1), \mathcal{B}((0,1)), \lambda \equiv \text{Leb})$, such that
$$\int \left(\sum_{n=1}^{\infty} f_n\right) d\lambda \neq \sum_{n=1}^{\infty} \int f_n d\lambda.$$

4. Let μ be a finite measure on $\mathbb{R}$ and $F(x) = \mu((-\infty, x])$. Show that
$$\int F(x+c) - F(x) dx = c\mu(\mathbb{R}).$$
Hint: Fubini's theorem.

5. Consider
$$F(x) = \begin{cases} 0 & x < 0 \\ 0.2 + x & 0 \le x < 0.2 \\ 0.4 + x & 0.2 \le x < 0.5 \\ 1 & x \ge 0.5 \end{cases}$$
This is a cumulative distribution function of a random variable X. Compute $\mathbb{E}X$ and $\mathbb{E}X^2$.

6. For any $\epsilon > 0$ and $\delta > 0$, find a random variable such that $\mathbb{E}X = 0, \text{Var}(X) = 1$, and $\mathbb{P}(|X| > \epsilon) < \delta$. It suffices to describe the random variable by its distribution.

   Hint: can you find an example of a random variable that takes only two different values ($\mathbb{P}(X \notin \{a,b\}) = 0$ for some a, b)? Such random variables are easy for calculation.

7. Let X be a random variable with $\mathbb{E}|X| < \infty$. Show that
$$\lim_{n\to\infty} \mathbb{E}(X\mathbf{1}_{\{|X|\le n\}}) = \mathbb{E}X \quad \text{and} \quad \lim_{n\to\infty} \mathbb{E}(X\mathbf{1}_{\{|X|>n\}}) = 0.$$
If instead of $\mathbb{E}|X| < \infty$ assume that X is non-negative, then do the above conclusions still hold? Justify your answer.

8. Consider a non-negative random variable X. Show that
$$\lim_{y\to\infty} y\mathbb{E}\left(\frac{1}{X}\mathbf{1}_{\{X>y\}}\right) = 0.$$