Due Wed Feb 10 in class.
Problems with ($*$) are required but no need to hand in.

1. ($*$) Read lecture notes 1.3 – 1.5, or corresponding materials from textbook.

2. Let f and g be two measurable functions in a common measurable space $(\Omega, \mathcal{F})$. Prove that $f+g$ is measurable, by applying Theorem 1.3.1 of textbook.

   Hint: first argue that by Theorem 1.3.1, it suffices to show for all $x \in \mathbb{R}$,
$$\{\omega \in \Omega : f(\omega) + g(\omega) < x\}$$
   is measurable. Express this set in terms of $\{\omega \in \Omega : f(\omega) < y\}$ and $\{\omega \in \Omega : g(\omega) < z\}$, for $y, z \in \mathbb{Q}$.

3. Consider the measure space $([0,1], \mathcal{B}([0,1]), \mathrm{Leb})$. Consider $f_n(x) = x^n, n \in \mathbb{N}$ and $f(x) = 0$.

   Is it true that $\lim_{n\to\infty} f_n = f$? Is it true that $\lim_{n\to\infty} f_n = f$ a.e.?

   Find another measure μ, such that for the same f_n and f but considered in the measure space $([0,1], \mathcal{B}([0,1]), \mu)$,
$$\mu\left(\left\{\omega \in [0,1] : \lim_{n\to\infty} f_n(\omega) = f(\omega)\right\}\right) = 0.$$

4. We have seen that Lebesgue integration includes Riemann integration and infinite series as special cases.

   State monotone convergence theorem for the special case of Riemann integration. State dominated convergence theorem for the special case of infinite series (taking $\Omega = \{\omega_n\}_{n\in\mathbb{N}}$ be any countable set, $\mathcal{F} = 2^\Omega$ and μ in the form of $\mu = \sum_{n=1}^\infty a_n \delta_{\omega_n}$).

5. Provide an example of a sequence of measurable functions $\{f_n\}_{n\in\mathbb{N}}$ defined on an appropriate measure space $(\Omega, \mathcal{F}, \mu)$, such that
$$\sup_{n\in\mathbb{N}} |f_n| < \infty, \quad \lim_{n\to\infty} f_n = f \quad \text{and} \quad \lim_{n\to\infty} \int f_n d\mu \neq \int f d\mu.$$

6. Provide an example of a sequence of measurable functions $\{f_n\}_{n\in\mathbb{N}}$ defined on an appropriate measure space $(\Omega, \mathcal{F}, \mu)$, such that
$$f_n \uparrow f \text{ as } n \to \infty \quad \text{and} \quad \lim_{n\to\infty} \int f_n d\mu \neq \int f d\mu.$$

7. Provide an example of a sequence of measurable functions $\{f_n\}_{n\in\mathbb{N}}$ defined on an appropriate measure space $(\Omega, \mathcal{F}, \mu)$, such that
$$\mu(\Omega) = 1, \quad \lim_{n\to\infty} f_n = 0 \quad \text{and} \quad \lim_{n\to\infty} \int f_n d\mu = 1.$$

8. Let $\{b_{m,n}\}_{m,n\in\mathbb{N}}$ be a collection of non-negative real numbers. Consider

$$S_n := \sum_{m\in\mathbb{N}} b_{m,n}, n \in \mathbb{N},$$

and the limit as $n \to \infty$. Assume that for each m, $\lim_{n\to\infty} b_{m,n} = \beta_m \in \mathbb{R}$ and $\sum_{m=1}^{\infty} \beta_m < \infty$.

(i) We are interested in $\liminf_{n\to\infty} S_n$. Formulate this problem as a special case of Fatou's lemma, by specifying the measure space $(\Omega, \mathcal{F}, \mu)$ and measurable functions $\{f_n\}_{n\in\mathbb{N}}$ and f. What does Fatou's lemma tell in this case?

Hint: consider $f(m) := \beta_m, m \in \mathbb{N}$.

(ii) By providing a counterexample, show that one should not expect $\lim_{n\to\infty} S_n$ to exist as a finite number, without further assumptions on $b_{m,n}$.

(iii) Provide an additional assumption on $b_{m,n}$ that guarantees

$$\lim_{n\to\infty} S_n = \sum_{m=1}^{\infty} \beta_m.$$