Name:

Due Wed Jan. 27 in class.
Problems with $(*)$ are required but no need to hand in.

1. Exercise 1.1.17 from lecture notes.

2. $(*)$ Read lecture notes Sections 1.1, or corresponding materials in textbook.

3. Let $p \in (0,1)$ be arbitrary. Construct 3 Bernoulli random variables X with parameter p, (that is, $\mathbb{P}(X = 1) = p = 1 - \mathbb{P}(X = 0)$) in three different probability spaces:

   (1) $((0,1), \mathcal{B}((0,1)), \mathrm{Leb})$,

   (2) $(\{0,1\}, 2^{\{0,1\}}, \mathbb{P})$ for some $\mathbb{P}$ to choose,

   (3) $([0,\infty), \mathcal{B}([0,\infty)), \mathbb{P})$ where $\mathbb{P}$ is determined by
   $$\mathbb{P}((a,b)) = \int_a^b e^{-x} dx, 0 \le a < b < \infty.$$
   Here, $([0,\infty), \mathcal{B}([0,\infty)))$ is defined similarly as $((0,1), \mathcal{B}((0,1)))$, and $\mathbb{P}$ corresponds to the standard exponential distribution.

   Express each random variable as a function explicitly.

4. Construct two different random variables in a common probability space, so that they have the same Bernoulli distribution with parameter $p \in (0,1)$. Namely, provide explicitly a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and two measurable functions $X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and $Y : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, such that the two random variables have the same Bernoulli distribution with parameter p, and $\mathbb{P}(\omega : X(\omega) \neq Y(\omega)) > 0$.

5. Consider the measurable space $((0,1), \mathcal{B}((0,1)))$, and two probability measures $\mathbb{P}_1 := \mathrm{Leb}$ and
   $$\mathbb{P}_2(A) := \sum_{n=1}^{\infty} \frac{1}{2^n} \mathbf{1}_{\{\frac{1}{n+1} \in A\}}, A \in \mathcal{B}((0,1)).$$
   Consider $X(\omega) := \omega, \omega \in (0,1)$.

   (a) (*) Why $\mathbb{P}_2$ is a probability measure? How would you describe it in words?

   (b) Compute $\mathbb{P}_1(X \ge 1/4)$ and $\mathbb{P}_2(X \ge 1/4)$.

   (c) Write the cumulative distribution functions of X in two probability spaces $((0,1), \mathcal{B}((0,1)), \mathbb{P}_1)$ and $((0,1), \mathcal{B}((0,1)), \mathbb{P}_2)$ respectively. Hint: for the second case, you may want to take a look at Example 1.2.6 in textbook.

6. Let X be a random variable with uniform $(0,1)$ distribution, defined in certain probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let Y be another random variable defined in the same probability space by $Y(\omega) = \max\{X(\omega), 1/3\}$. Describe the distribution μ_Y of Y in the form of

$$\mu_Y = \mu_1 + \mu_2$$

where μ_1 and μ_2 are discrete and continuous measures, respectively.