Due Monday March 2 in class.
Problems with $(*)$ are required but no need to hand in.

1. Exercises 1.7.2, 1.7.3, 1.7.4 $(*)$, 1.7.5 $(*)$.

2. Let $\{b_{m,n}\}_{m,n\in\mathbb{N}}$ be a collection of non-negative real numbers. Consider

$$S_n := \sum_{m=1}^{\infty} b_{m,n}, n \in \mathbb{N},$$

and the limit as $n \to \infty$. Assume that for each m, $\lim_{n\to\infty} b_{m,n} = \beta_m \in \mathbb{R}$ and $\sum_{m=1}^{\infty} \beta_m < \infty$.

(i) Consider $(\Omega, \mathcal{F}, \mu) = (\mathbb{N}, 2^{\mathbb{N}}, \lambda)$ where λ is the counting measure on $\mathbb{N}$ (i.e., $\lambda(\{k\}) = 1, k \in \mathbb{N}$). Then for $m \in \mathbb{N}$, introduce

$$f_n(m) := b_{m,n}, m, n \in \mathbb{N}.$$

In this way, for each $n \in \mathbb{N}$, f_n is a measurable function on $(\mathbb{N}, 2^{\mathbb{N}}, \lambda)$. Justify first that

$$\sum_{m=1}^{\infty} b_{m,n} = \int_{\mathbb{N}} f_n d\lambda, n \in \mathbb{N}.$$

(ii) What does Fatou's lemma tell in this case?

(iii) By providing a counterexample, show that one should not expect

$$\lim_{n\to\infty} S_n = \sum_{m=1}^{\infty} \beta_m, \tag{1}$$

without further assumptions on $b_{m,n}$.

(iv) In order to apply monotone convergence theorem to obtain (1), what additional assumption do we need?