Due Wed Feb. 11 in class.
Problems with (∗) are required but no need to hand in.
Solution to Exercise 1.3.1 is on page 2.

1. (∗) Read Chapter 1.4, 1.5.

2. Exercises 1.4.1, 1.4.2 (∗).

3. In step 3 of the definition of Lebesgue integral, we introduce, for measurable function f such that $f \geq 0$ μ-almost everywhere,

$$\int f d\mu := \sup\Big\{\int h d\mu : 0 \leq h \leq f, \mu\text{-a.e.},$$
$$h \text{ measurable and bounded}, \mu\{\omega : h(\omega) > 0\} < \infty\Big\}.$$

Provide an example of $f \geq 0$ and $\int f d\mu = \infty$, using this definition.

4. Lebesgue integral and Riemann integral. We compare the two definitions of integrals. We have seen Lebesgue integrals in class, in form of $\int f(x)\mu(dx)$, and here we focus on the case that the measure space with respect to which the Lebesgue integral is defined is $((0,1), \mathcal{B}((0,1)), \mu = \text{Leb})$. In this way, we write $\int f(x)\mu(dx) = \int_{[0,1]} f(x)dx$.

   (a) Recall the definition of Riemann integral for $\int_0^1 f(x)dx$, for continuous function f.

   (b) (∗) Show that for all non-negative continuous functions $f(x)$ on $[0,1]$, the two integrals have the same value.

5. (a) Construct a sequence of measurable functions $\{f_n\}_{n\in\mathbb{N}}$ on $[0,1]$ that converge to 0 in measure (w.r.t. the Lebesgue measure). That is,

$$\lim_{n\to\infty} \text{Leb}(\{\omega \in (0,1) : |f_n(\omega)| > \epsilon\}) = 0 \text{ for all } \epsilon > 0. \quad (1)$$

   Draw a picture of $f_n(x)$.

   (b) Prove that (1) is implied by the following condition

$$\lim_{n\to\infty} f_n(x) = 0, \text{ for almost all } x \in [0,1]. \quad (2)$$

   (c) Does your example satisfy (2)? If yes, can you construct an example that satisfies (1) but not (2)? Draw a picture if you can.

   (d) Do you have

$$\lim_{n\to\infty} \int |f_n(x)|dx = 0, \quad (3)$$

   for your example(s)? Are conditions (3) and (1) equivalent? Justify your answer.

   (e) (∗) When viewing all the f_n's as random variables, what can you say about their distributions?

6. In Theorem 1.5.4, it is assumed that $f_n \geq 0$. Show that this condition cannot be removed by constructing a counterexample.

Solution to Exercise 1.3.1. *Proof:* The goal is to show

$$\sigma(X^{-1}(\mathcal{A})) = \sigma(X).$$

It is easy to show $\sigma(X)$ is a σ-algebra. (The proof is omitted.)

We first show $\sigma(X^{-1}(\mathcal{A})) \subset \sigma(X)$. To see this, it suffices to observe that $\sigma(X) = \{\{X \in B\} : B \in \mathcal{S}\}$, which contains $X^{-1}(\mathcal{A})$ by definition, and recall the definition that $\sigma(X^{-1}(\mathcal{A}))$ is the smallest σ-algebra containing $X^{-1}(\mathcal{A})$.

Now, since we know $\sigma(X^{-1}(\mathcal{A})) \subset \sigma(X)$, there exists a collection of sets $\mathcal{F} \subset \mathcal{S}$ such that

$$\sigma(X^{-1}(\mathcal{A})) = \{\{X \in B\} : B \in \mathcal{F}\}.$$

Since $\sigma(X^{-1}(\mathcal{A}))$ is a σ-algebra, one can show that $\mathcal{F}$ is also a σ-algebra. (The proof is omitted.) But $\mathcal{S}$ is the smallest σ-algebra containing $\mathcal{A}$. Therefore it follows that $\mathcal{F} = \mathcal{S}$. We have thus proved the desired result.