Name

1. On Halloween, Mr. and Mrs. Smith gave out 100 (indistinguishable) candies to 10 kids from the neighbor area. It might happen that a kid came too late so that the candies had been all gone already.

   (a) How many different ways of distribution could have occurred?
   $\binom{109}{100} = \binom{109}{9}$.

   (b) What about if Mr. and Mrs. had planned well in advance to guarantee that every kid would have at least one candy?
   $\binom{99}{9}$.

   (c) Describe the two problems in the balls and bars language.

   Part (a) corresponds to the case of distributing n $(= 100)$ balls into r $(r = 10)$ groups with empty group allowed. So the problem becomes how many ways to arrange $r - 1$ undistinguished bars and n undistinguished balls in a row.

   Part (b) corresponds to the case of distributing n balls into r non-empty groups. So the problem comes how many ways to choose $r - 1$ positions out of $n - 1$ ones.

2. Let X be a random variable satisfying $\mathbb{P}(X > x) = x^{-\alpha}$ for $x > 1$ for some $\alpha > 0$. In this case $X > 1$. Find the c.d.f. and p.d.f. of $(2X)^2$.

   We first compute the c.d.f. of X.

$$\mathbb{P}(X \le x) = 1 - x^{-\alpha}, x > 1.$$

   Next, we compute the c.d.f. of $Y = (2X)^2$. Before that, remark that since $X > 1, Y = (2X)^2 > 4$.

$$\begin{aligned}\mathbb{P}(Y \le y) &= \mathbb{P}((2X)^2 \le y) \\ &= \mathbb{P}(X \le y^{1/2}/2) = 1 - (y^{1/2}/2)^{-\alpha} = 1 - 2^{\alpha} y^{-\alpha/2}, y > 4.\end{aligned}$$

   At last, we compute the p.d.f. of Y.

$$p_Y(y) = \frac{d}{dy}\mathbb{P}(Y \le y) = \left(1 - 2^{\alpha} y^{-\alpha/2}\right)' = \alpha 2^{\alpha-1} y^{-\alpha/2-1}, y > 4.$$

3. Choose independently two numbers B and C at random from the interval $[0, 1]$ with uniform density. Note that the point (B, C) is then chosen at random in the unit square.

(a) Compute $\mathbb{P}(\min(B, C) > 1/5)$.

$$\mathbb{P}(\min(B, C) > 1/5) = \mathbb{P}(B > 1/5 \text{ and } C > 1/5)$$
$$= \mathbb{P}(B > 1/5)\mathbb{P}(C > 1/5) = \left(\frac{4}{5}\right)^2.$$

(b) Find the probability density function of $Z = \max(B, C)$.

$$\mathbb{P}(Z \leq z) = \mathbb{P}(\max(B, C) \leq z) = \mathbb{P}(B \leq z \text{ and } C \leq z)$$
$$\mathbb{P}(B \leq z)\mathbb{P}(C \leq z) = z^2, z \in [0, 1].$$

The p.d.f. of Z thus equals

$$p_Z(z) = (z^2)' = 2z, z \in (0, 1),$$

and zero elsewhere.

(c) Compute $\mathbb{E}Z$ and $\mathrm{Var}Z$.

$$\mathbb{E}Z = \int_0^1 z(2z)dz = 2\int_0^1 z^2 dz = \frac{2}{3}.$$
$$\mathbb{E}(Z^2) = \int_0^1 z^2(2z)dz = 2\int_0^1 z^3 dz = \frac{1}{2}.$$
$$\mathrm{Var}(Z) = \mathbb{E}(Z^2) - (\mathbb{E}Z)^2 = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}.$$

(d) Compute $\mathbb{E}(2B + C)$ and $\mathrm{Var}(2B + C)$.

Remark that B and C have the same distribution, thus

$$\mathbb{E}(2B + C) = 2\mathbb{E}B + \mathbb{E}C = 3\mathbb{E}B = \frac{3}{2}.$$

For the variance, remark that B and C are independent. Then,

$$\mathrm{Var}(2B + C) = \mathrm{Var}(2B) + \mathrm{Var}(C) = 4\mathrm{Var}(B) + \mathrm{Var}(C) = 5\mathrm{Var}(B).$$

Since

$$\mathrm{Var}(B) = \int_0^1 (x - 1/2)^2 dx = \int_{-1/2}^{1/2} x^2 dx = 2\int_0^{1/2} x^2 dx = \frac{1}{12},$$

$\mathrm{Var}(2B + C) = 5\mathrm{Var}(B) = 5/12$.

4. Let X be a random variable with probability density function

$$p_X(x) = \begin{cases} cx^2 & x \in [0,1] \\ 0 & \text{otherwise.} \end{cases}$$

(a) Identify the constant c.

The constant c is such that $\int_0^1 cx^2 dx = 1$, whence $c = 3$.

(b) Compute $\mathbb{E}X$ and $\mathrm{Var}X$.

$$\mathbb{E}X = \int_0^1 x \cdot 3x^2 dx = 3\int_0^1 x^3 dx = \frac{3}{4}.$$

$$\mathbb{E}(X^2) = \int_0^1 x^2 \cdot 3x^2 dx = 3\int_0^1 x^4 dx = \frac{3}{5}.$$

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}X)^2 = \frac{3}{5} - \frac{9}{16} = \frac{3}{80}.$$

5. A card is drawn at random from a deck consisting of cards numbered 1 through 9. A player wins 1 dollar if the number on the card is odd and loses 1 dollar if the number if even. What are the expected value and variance of his winnings?

Let X be the random variable representing the winning of the gambler. Then, $\mathbb{P}(X = 1) = 5/9$ and $\mathbb{P}(X = -1) = 4/9$.

$$\mathbb{E}X = 1 \cdot \frac{5}{9} + (-1) \cdot \frac{4}{9} = \frac{1}{9}.$$

$$\mathbb{E}(X^2) = 1^2 \cdot \frac{5}{9} + (-1)^2 \cdot \frac{4}{9} = 1.$$

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}X)^2 = 1 - \left(\frac{1}{9}\right)^2 = \frac{80}{81}.$$

6. Throw a die three times.

   (a) What is the probability that the sum of the first two throws is strictly larger than 8?

   Write $S = X_1 + X_2$ with X_1 and X_2 representing the numbers from the first two throws. Then X_1 and X_2 are independent with discrete uniform distribution over $\{1, \ldots, 6\}$. We count the total number, say T, of outcomes of the first two throws that have sum strictly larger than 8. By enumeration, $T = 4 + 3 + 2 + 1 = 10$. (For example, the total number of outcomes that sum to 9 is 4: $(3, 6), (4, 5), (5, 4), (6, 3)$.)

   Since all 36 outcomes are equally likely, $\mathbb{P}(S > 8) = 10/36 = 5/18$.

   (b) What is the probability that the sum of the three throws is 11?

   We count all the outcomes that have sum 11. First, observe that all the collection of three integers from $\{1, \ldots, 6\}$ that sum to 11 consists of (regardless of ordering)

$$(1, 4, 6), (1, 5, 5), (2, 3, 6), (2, 4, 5), (3, 3, 5), (3, 4, 4).$$

   Next, we take into account the different ordering for each case. The corresponding numbers of different orderings are $6, 3, 6, 6, 3, 3$. Therefore, the total number of ordered outcomes that add to 11 equals

$$6 + 3 + 6 + 6 + 3 + 3 = 27.$$

   The desired probability then equals $27/6^3 = 1/8$.