Name

The total time for this quiz is 55 minutes. Show steps clearly.

1. We are interested in how to schedule all the regular season games of an NBA team, say San Antonio Spurs, the defending champion. The Spurs have 82 regular season games, with 41 home games and 41 road games. To simplify our consideration, we do not take into account the opponent team that each time the Spurs are playing against. As a fact, Spurs are playing on the road against Lakers this Friday (Nov 14), and before that, they have played 3 home games and 4 road games this season.

   How many different schedules the Spurs may have, so that right before playing the 5th road game, they have exactly 3 home games? (Again, no consideration on the opponent teams, nor the game dates.)

   Solution: Before playing the 5th road game, there are exactly 3 home games. So the 5th road game is the 8th game of the season, and there are 7 games before that.

   The question is asking how many different ways to arrange 4 road games and 3 home games among 7 games. The answer is $\binom{7}{4}$.

2. Let X be a uniform random variable on the interval $(0, 2)$.

   (a) What are the p.d.f. and c.d.f. of X?

   Solution: The p.d.f. of X is $p_X(x) = 1/2$ for $x \in (0, 2)$, and $p_X(x) = 0$ otherwise.

   The c.d.f. of X equals

   $$F_X(x) = \int_0^x p_X(y)dy = \frac{1}{2}x, x \in [0, 2], \tag{1}$$

   $F_X(x) = 0, x \leq 0$ and $F_X(x) = 1, x \geq 2$.

   (b) What are the p.d.f. and c.d.f. of $Y = 1 - X/2$?

   Solutions: We first compute the c.d.f.

   $$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(1 - X/2 \leq y).$$

   Remark that $\{1-X/2 \leq y\} = \{X \geq 2(1-y)\}$. Thus, the above equals

   $$\mathbb{P}(X \geq 2(1 - y)) = 1 - \mathbb{P}(X \leq 2(1 - y)).$$

   Now by (1),

   $$F_Y(y) = 1 - F_Y(2(1 - y)) = 1 - (1 - y) = y, y \in [0, 1]. \tag{2}$$

   It then follows that $p_Y(y) = 1, y \in [0, 1]$. One can also easily verify that $p_Y(y) = 0$ elsewhere. In fact, (2) tells that Y is a uniform random variable on interval $(0, 1)$.

3. Suppose we have a die that is not fair; instead, at each throw it shows $1, \ldots, 5$ with probability 0.1 each, and 6 with probability 0.5.

   (a) What is the mean and variance of the value of this die (one throw)?

   Solutions: Let X denote the value of the first throw of the die.
   $\mathbb{E}X = 1 \times 0.1 + 2 \times 0.1 + 3 \times 0.1 + 4 \times 0.1 + 5 \times 0.1 + 6 \times 0.5 = 4.5.$
   $\mathbb{E}X^2 = (1^2 + 2^2 + 3^2 + 4^2 + 5^2) \times 0.1 + 6^2 \times 0.5 = 23.5.$
   $\mathrm{Var}(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2 = 23.5 - 4.5^2 = 3.25.$

   (b) Throw this die twice and let S denote the sum of the two values. What is the mean and variance of S?

   Solutions: Let X_1 and X_2 denote the values of the two throws, respectively. Then $S = X_1 + X_2$.

   $$\mathbb{E}S = \mathbb{E}(X_1 + X_2) = \mathbb{E}X_1 + \mathbb{E}X_2 = 2\mathbb{E}X = 9.$$

   Since X_1 and X_2 are independent,

   $$\mathrm{Var}(S) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) = 2\mathrm{Var}(X) = 6.5.$$

   (c) Throw this die twice. What is the probability that the sum is 10?

   Solutions: To have a sum 10, there are three possibilities: $\{4,6\}, \{5,5\}, \{6,4\}$. Thus,

   $$\begin{aligned} \mathbb{P}(S = 10) &= \mathbb{P}(X_1 = 4, X_2 = 6) + \mathbb{P}(X_1 = 5, X_2 = 5) \\ &\quad + \mathbb{P}(X_1 = 6, X_2 = 4) \\ &= 0.1 \times 0.5 + 0.1 \times 0.1 + 0.5 \times 0.1 = 0.11, \end{aligned}$$

   where in the second equality above we used the fact that X_1 and X_2 are independent.

4. Consider two independent random variables B and C, both taking values in $[0,1]$, with the following p.d.f.s:

$$p_B(x) = 1, p_C(x) = 2x, x \in [0,1].$$

(a) Find the p.d.f. of $Y = \max(B, C)$.

Solution: We first compute the c.d.f. of Y.

$$\begin{aligned}\mathbb{P}(Y \leq y) = \mathbb{P}(\max(B,C) \leq y) &= \mathbb{P}(B \leq y)\mathbb{P}(C \leq y) \\ &= \int_0^y dy \times \int_0^y 2x dx = y^3, y \in [0,1].\end{aligned}$$

Therefore, $p_Y(y) = d/dy(\mathbb{P}(Y \leq y)) = 3y^2, y \in [0,1]$ and $p_Y(y) = 0$ elsewhere.

(b) Find the p.d.f. of $Z = B + C$.

Hint: first determine the range of Z. You can use the formula

$$p_Z(z) = \int_{-\infty}^{\infty} p_B(z-x)p_C(x)dx.$$

Solution: By formula, since $p_C(x)$ is nonzero only on $x \in (0,1]$,

$$p_Z(z) = \int_0^1 p_B(z-x)2xdx.$$

By change of variables ($u = z - x$),

$$p_Z(z) = \int_{z-1}^{z} p_B(u)2(z-u)du.$$

Since p_B is nonzero only on $[0,1]$, we observe that $p_Z(z)$ is zero if $z - 1 > 1$ or $z < 0$. That is,

$$p_Z(z) = 0, z \notin (0,2). \tag{3}$$

Next, Discuss the case $z \in (0,1]$ and $z \in [1,2)$ respectively. When $z \in (0,1]$, since $p_B(u) = 0$ for $u \notin (0,1)$, we have

$$\begin{aligned}p_Z(z) &= \int_0^z 2(z-u)du \\ &= \int_0^z 2zdu - \int_0^z 2udu = 2z^2 - z^2 = z^2, z \in (0,1]. \end{aligned} \tag{4}$$

Similarly, for $z \in [1,2)$,

$$\begin{aligned}p_Z(z) &= \int_{z-1}^1 2(z-u)du = \int_{z-1}^1 2zdu - \int_{z-1}^1 2udu \\ &= 2z(2-z) - (1-(z-1)^2) = 2z - z^2, z \in [1,2). \end{aligned} \tag{5}$$

The p.d.f. p_Z is thus given by (3), (4) and (5). (To convince oneself that p_Z is a p.d.f., one can verify here that $\int_0^2 p_Z(z)dz = 1$.)