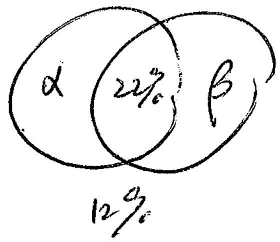
Name

There are 6 problems, total 30 points. Present your solutions clearly.

You may provide formulae using factorials ($n!$) and binomial coefficients ($\binom{n}{j}$) as solutions. No need to simplify nor to calculate the expressions in such a case.

1. (3 points) Among a large group of patients recovering from shoulder injuries, it is found that 22% visit both a physical therapist and a chiropractor, whereas 12% visit neither of these. The probability that a patient visits a chiropractor exceeds by 0.14 the probability that a patient visits a physical therapist.

   Determine the probability that a randomly chosen member of this group visits a physical therapist.



$\alpha + \beta = 66\%.$

$\alpha - \beta = 14\%$

$\Rightarrow \alpha = 40\% \quad \beta = 26\%$

$22\% + 26\% = 48\%$

2. (3 points) Suppose X is a random variable satisfying $\mathbb{P}(X > x) = x^{-\alpha}$ for $x > 1$, with some fixed $\alpha > 0$. Observe that $X > 1$. Find the c.d.f. and p.d.f. of random variable $Y = X^\beta$, for some fixed $\beta > 0$. Show all steps.

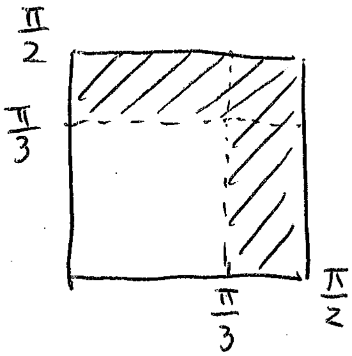
CDF: $\mathbb{P}(Y \le y) = \mathbb{P}(X^\beta \le y) = \mathbb{P}(X \le y^{1/\beta})$

$= 1 - y^{-\alpha/\beta}. \quad y > 1. \quad (0, \text{ o.w}).$

pdf: $f_Y(y) = \frac{d}{dy} \mathbb{P}(Y \le y) = \begin{cases} \frac{\alpha}{\beta} y^{-\alpha/\beta - 1} & y > 1 \\ 0 & \text{o.w.} \end{cases}$

3. (6 points) Choose independently two numbers B and C at random from the interval $[0, \pi/2]$, with the following prescribed probability density functions f_B and f_C respectively. Throughout, assume $f_B(x) = f_C(x) = 0$ for $x \notin [0, \pi/2]$. Find the probability that $\max(B, C) > \pi/3$ in each case.

(a) $f_B(x) = f_C(x)$ are constant functions for $x \in [0, \pi/2]$.



$$\mathbb{P}(\max(B,C) > \tfrac{\pi}{3}) = \frac{\text{area of } \boxed{\,}}{(\frac{\pi}{2})^2}$$

$$= 1 - \frac{(\frac{\pi}{3})^2}{(\frac{\pi}{2})^2} = \frac{5}{9}$$

(b) $f_B(x) = \sin x$, $f_C(x) = \cos x$.

~~$\{B > \frac{\pi}{3}\}$~~

Same pic above

$$\mathbb{P}(\max(B \cdot C) > \tfrac{\pi}{3}) = \iint_{\boxed{\,}} \sin x \cos y \, dy \, dx$$

$$= \int_0^{\frac{\pi}{2}} \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x \cos y \, dy \, dx + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \int_0^{\frac{\pi}{3}} \sin x \cos y \, dy \, dx$$

$$= \int_0^{\frac{\pi}{2}} \sin x \, dx \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos y \, dy + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x \, dx \int_0^{\frac{\pi}{3}} \cos y \, dy$$

$$= \left(1 - \frac{\sqrt{3}}{2}\right) + \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = 1 - \frac{\sqrt{3}}{4}.$$

Or, alternatively

$$\mathbb{P}(\max(B \cdot C) > \tfrac{\pi}{3})$$
$$= 1 - \mathbb{P}(\max(B \cdot C) \leq \tfrac{\pi}{3})$$
$$= 1 - \mathbb{P}(B \leq \tfrac{\pi}{3})\, \mathbb{P}(C \leq \tfrac{\pi}{3})$$
$$= 1 - \int_0^{\frac{\pi}{3}} \sin x \, dx \int_0^{\frac{\pi}{3}} \cos y \, dy$$
$$= 1 - \frac{\sqrt{3}}{4}$$

4. (3 points) In a bridge game of 4 players, Rachel and Ross are partners. Each player has 13 cards from a randomly shuffled deck of 52 cards (so that all possible hands of cards have equal probability).

What is the conditional probability that, given that Rachel has exactly 4 cards of hearts, Ross also has exactly 4 cards of hearts?

total nb. of hands for Rachel and Ross. each having exactly 4 hearts

$$\#1 = \underbrace{\binom{13}{4}}_{\text{hearts}} \cdot \underbrace{\binom{39}{9}}_{\text{no hearts}} \cdot \underbrace{\binom{13-4}{4}}_{\text{hearts}} \cdot \underbrace{\binom{39-9}{9}}_{\text{no hearts}}$$

Rachel Ross

total nb. of hands for Rachel having exact 4 cards

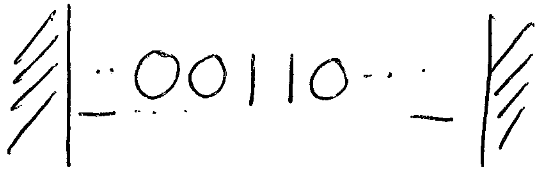
$$\#2 = \binom{13}{4} \cdot \binom{39}{9} \cdot \binom{39}{13}$$

The desired cond. proba

$$= \frac{\#1}{\#2} = \frac{\binom{9}{4}\binom{30}{9}}{\binom{39}{13}}$$

5. (9 points) Suppose one wants to put n indistinguishable balls into j distinguishable boxes, with $j \leq n$.

(a) If it is allowed to have empty boxes at the end, how many different ways can this be done?



$n+j-1$ positions

n balls and $j-1$ bars to put in

Consecutive bars are allowed (why?).

$$\binom{n+j-1}{j-1} = \binom{n+j-1}{n}$$

(b) If in addition it is required to have at least one ball in each box, how many different ways can this be done?

~~n~~ ~~j-1~~ $n-1$ possible pos. to put $j-1$ bars

each pos. at most one bar

$$\binom{n-1}{j-1}$$

(c) Keep flip a coin until one sees j heads. What is the probability that when the j-th heads shows up, there are already n tails in previous tossing?

Hint: Which model in (a) and (b) is closely related to this problem?

$$T\,T \cdots \underset{\substack{\uparrow \\ \text{first } H}}{T\,H}\,T \cdots \underset{\substack{\uparrow \\ \text{second } H}}{T\,H}\,T \cdots \underset{\substack{\uparrow \\ j\text{th } H}}{T\,H}$$

View Hs as bars and Ts as ~~tails~~ balls.

Consecutive bars correspond to consecutive heads (allowed)

So apply model (a): $\binom{n+j-1}{j-1} = \binom{n+j-1}{n}$ (# of patterns) all w/ equal proba.

$$\text{proba} = \binom{n+j-1}{j-1} \cdot \left(\frac{1}{2}\right)^{n+j}$$

6. (6 points) The Yankees are playing the Dodgers in a world series. The Yankees win each game with probability 0.6. The series is won by the first team to win four games.

(a) What is the probability that Yankees win the series in game 5 (with the final score 4:1)?

How many different results leading to 4:1?

LWWWW. WLWWW ... WWWLW # $\binom{4}{1}$.

Each scenario happens w/ proba $0.6^4\ 0.4$

So proba $= \binom{4}{1} \cdot 0.6^4 \cdot 0.4$

(b) What is the probability that the Yankees win the series?

Discuss by case: how many losses before final win? (4:ℓ).

$$\sum_{\ell=0}^{3} \binom{3+\ell}{\ell} \cdot 0.6^4\ 0.4^\ell$$

↑

\# of scenarios w/ ℓ losses before final w (4:ℓ).