- 6.3.2

(a) Observe that the p.d.f. is symmetric with respect to zero ($f_X(x) = f_X(-x)$), thus the random variable has zero mean. For the variance,

$$\begin{aligned}\mathrm{Var}(X) = \mathbb{E}(X^2) &= \int_{-1}^{1} x^2 \frac{3}{4}(1-x^2)dx \\ &= \int_{-1}^{1} \frac{3}{4}x^2 dx - \int_{-1}^{1} \frac{3}{4}x^4 dx = \frac{1}{2} - \frac{3}{10} = \frac{1}{5}.\end{aligned}$$

(b) By symmetry, $\mathbb{E}X = 0$. For the variance, again by symmetry,

$$\mathrm{Var}(X) = \int_{-1}^{1} x^2 \frac{\pi}{4}\cos(\pi x/2)dx = 2\int_{0}^{1} x^2 \frac{\pi}{4}\cos(\pi x/2)dx. \tag{1}$$

By integration by parts,

$$\begin{aligned}\int_0^1 x^2\frac{\pi}{2}\cos(\pi x/2)dx &= x^2\sin(\pi x/2)\Big|_0^1 - \int_0^1 2x\sin(\pi x/2)dx \\ &= 1 - \int_0^1 2x\sin(\pi x/2)dx. \qquad (2)\end{aligned}$$

Again by integration by parts,

$$\begin{aligned}-\int_0^1 2x\sin(\pi x/2)dx &= 2x\frac{2}{\pi}\cos(\pi x/2)\Big|_0^1 - \int_0^1 \frac{4}{\pi}\cos(\pi x/2)dx \\ &= -\frac{4}{\pi}\int_0^1 \cos(\pi x/2)dx = -\frac{8}{\pi^2}. \qquad (3)\end{aligned}$$

To sum up, combining (1), (2) and (3) we obtain $\mathrm{Var}(X) = 1 + 8/\pi^2$.

- 6.3.10

(c) By definition,

$$\begin{aligned}\mathbb{E}(\min(X,Y)) &= \int_0^1\int_0^1 \min(x,y)dydx \\ &= \int_0^1\int_0^x ydydx + \int_0^1\int_x^1 xdydx \\ &= \int_0^1 \frac{x^2}{2}dx + \int_0^1 x(1-x)dx \\ &= \int_0^1 xdx - \int_0^1 \frac{x^2}{2}dx = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}.\end{aligned}$$

One can also use the fact that $\mathbb{E}\min(X,Y) + \mathbb{E}\max(X,Y) = \mathbb{E}(\min(X,Y) + \min(X < Y)) = \mathbb{E}(X+Y) = 1$, and the result we have seen in class that $\mathbb{E}\max(X,Y) = 2/3$. An alternative way to compute $\mathbb{E}\max(X,Y)$ is to first find the p.d.f. of $\max(X,Y)$, which equals $2x$ on $(0,1)$ and 0 elsewhere.

(e) $\mathbb{E}[(X+Y)^2] = \mathbb{E}(X^2 + 2XY + Y^2)$. Since X and Y have the same distribution and are independent,

$$\mathbb{E}[(X+Y)^2] = 2\mathbb{E}(X^2) + 2(\mathbb{E}X)^2 = 2 \cdot \frac{1}{3} + 2 \cdot \left(\frac{1}{2}\right)^2 = \frac{7}{6}.$$

- 7.1.2 Let X_1 and X_2 represent the stock price change for the first two days. Then the two random variables have the same distribution, and they are independent. Thus, for $S = X_1 + X_2$, it takes values from $\{-2, -1, 0, 1, 2, 3, 4\}$. The distribution of S equals

$$p_S = \begin{pmatrix} -2 & -1 & 0 & 1 & 2 & 3 & 4 \\ \frac{1}{16} & \frac{1}{4} & \frac{5}{16} & \frac{3}{16} & \frac{9}{64} & \frac{1}{32} & \frac{1}{64} \end{pmatrix}.$$

- 7.1.6

  (a) T_r is the negative binomial distribution with parameter $(r, 1-p)$. More precisely,

$$\mathbb{P}(T_r = k) = \binom{k+r-1}{k-1}(1-p)^k p^r, k = 0, 1, \ldots.$$

  (b) C_r is a binomial distribution with parameter (r, p).

  (c) We are asked to compute $\mathbb{E}C_r$ and $\mathrm{Var}C_r$. By knowledge of binomial distribution, $\mathbb{E}C_r = rp$ and $\mathrm{Var}C_r = rp(1-p)$.

- 7.2.2

  (b) By convolution,

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)dx = \int_3^5 \frac{1}{2} f_Y(z-x)dx.$$

  By change of variables,

$$f_Z(z) = \int_{z-5}^{z-3} \frac{1}{2} f_Y(y)dy.$$

  Observe that for $z < 6$ or $z > 10$, $(z-5, z-3) \cap (3,5) = \emptyset$. That is, $f_Z(z) = 0$ for $z \notin (6, 10)$. Now,

$$f_Z(z) = \int_3^{z-3} \frac{1}{4} dy = \frac{z-6}{4}, z \in (6, 8)$$

  and

$$f_Z(z) = \int_{z-5}^{5} \frac{1}{4} dy = \frac{10-z}{4}, z \in [8, 10).$$

  (c) One can repeat a similar argument as above. Or, a better way is the following. Let X' and Y' be the random variables considered in part (c), and let $Z' = X' + Y'$. Observe that one can write $X' = X - 2$ and $Y' = Y$, where X and Y are the random variables considered in part (b). Therefore, the random variable Z' considered in part (c) is nothing but a shift of Z in part (b) by -2. More precisely, $Z' = Z - 2$ and

$$f_{Z'}(z) = f_Z(z+2) = \begin{cases} (z-4)/4 & z \in (4,6) \\ (8-z)/4 & z \in [6,8) \\ 0 & \text{otherwise.} \end{cases}$$