Final exam: Dec. 8, Mon, 1:30-3:30pm. There will be 8 problems in total. Below are a few sample problems.

1. State the following definitions and results.

   Distribution function of a discrete random variable.

   Cumulative definition function of a continuous random variable.

   Probability density function of a continuous random variable.

   Chebychev inequality.

   Theorem of Law of Large Numbers.

   Central Limit Theorem.

2. An insurance company examines its pool of auto insurance customers and gathers the following information:

   (i) All customers insure at least one car.

   (ii) 70% of the customers insure more than one car.

   (iii) 20% of the customers insure a sports car.

   (iv) Of those customers who insure more than one car, 15% insure a sports car.

   Calculate the probability that a randomly selected customer insures exactly one car and that car is not a sports car.

   Solution: Percentage of customers having only one car: $1 - 70\% = 30\%$. Percentage of customers having only only a sports car and no other cars: $20\% - 70\% \times 15\% = 9.5\%$. Percentage of customers having only one car and no sports car: $30\% - 9.5\% = 20.5\%$.

3. (From Quiz 2) Let X be a uniform random variable on the interval $(0, 2)$.

   (a) What are the p.d.f. and c.d.f. of X?

   (b) What are the p.d.f. and c.d.f. of $Y = 1 - X/2$?

4. A player throws a dart to a disk-shaped target. The disk has radius 20 cm. Suppose that he always hits the target, and the position of the dart is uniformly distributed on the disk.

   (a) What is the probability that the dart hits within 5 cm of the center, and in the upper half of the disk?

   Solution:
   $$\frac{\frac{1}{2}\pi \times 5^2}{\pi \times 20^2} = \frac{1}{32}.$$

   (b) If it is known that the dart its the upper half of the disk, what is the conditional probability that the dart hits within 5 cm of the center?

   Solution:
   $$\frac{\frac{1}{2}\pi \times 5^2}{\frac{1}{2}\pi \times 20^2} = \frac{1}{16}.$$

5. Draw 5 cards from a randomly shuffled deck of 52 cards (so that all hands are equal likely).

   (a) What is the probability of having two As?

   Solution: $\dfrac{\binom{4}{2}\binom{48}{3}}{\binom{52}{5}}$.

   (b) What is the probability of having two As and three kings?

   Solution: $\dfrac{\binom{4}{2}\binom{4}{3}}{\binom{52}{5}}$.

   (c) What is the probability of having three As, conditioning on having at least two As?

   Solution: $\dfrac{\binom{4}{3}\binom{48}{2}}{\binom{4}{2}\binom{48}{3}+\binom{4}{3}\binom{48}{2}+\binom{4}{4}\binom{48}{1}}$.

6. Two persons draw 5 cards each from a randomly shuffled deck of 52 cards.

   (a) What is the probability that they have 3 As together?

   Solution: $\dfrac{2\binom{48}{5}\binom{4}{3}\binom{43}{2}+2\binom{4}{1}\binom{48}{4}\binom{3}{2}\binom{44}{3}}{\binom{52}{5}\binom{47}{5}}$.

   (b) What is the probability that the second person has 2 As, conditioning on the event that the first person has one A?

   Solution: $\dfrac{\binom{3}{2}\binom{44}{3}}{\binom{47}{5}}$.

7. In the gym, a group of 10 students are about to play a basketball game. How many different ways to divide them into two groups of 5 each?

   Solution: $\dfrac{\binom{10}{5}}{2}$.

8. How many different ways to distribute n indistinguishable cookies to r kids? How many ways to do so so that each kid has at least one?

9. Let X be a random variable with p.d.f. $p_X(x)$ proportional to x^{-4} for $x > 1$, and $p_X(x) = 0$ for $x \leq 1$.

   (a) Compute $\mathbb{E}X$ and $\mathrm{Var}X$.

   Solution: First identify the p.d.f. $p_X(x) = cx^{-4}, x \geq 1$ for some c. Since $\int_1^\infty cx^{-4}dx = 1$, it follows that $c = 3$.

   (b) Compute $\mathbb{P}(X > 10)$ and $\mathbb{P}(X > 20 \mid X > 10)$.

   Solution: $\mathbb{P}(X > 10) = \int_{10}^\infty 3x^{-4}dx = 10^{-3}$.

   $$\mathbb{P}(X > 20 \mid X > 10) = \frac{\mathbb{P}(X > 20)}{\mathbb{P}(X > 10)} = \frac{20^{-3}}{10^{-3}} = \frac{1}{8}.$$

10. A random variable X has c.d.f.

$$F_X(x) = \begin{cases} 0 & \text{for } x < 1 \\ \dfrac{x^2 - 2x + 2}{2} & \text{for } 1 \le x \le 2 \\ 1 & \text{for } x > 2 \end{cases}$$

Compute $\mathbb{E}X$ and $\mathrm{Var}X$.

Solution: First find p.d.f. by differentiation. This leads to

$$p_X(x) = 2x - 2, x \in [1, 2],$$

and $p_X(x) = 0$ elsewhere.

11. (From Quiz 2) Consider two independent random variables B and C, both taking values in $[0, 1]$, with the following p.d.f.s:

$$p_B(x) = 1, p_C(x) = 2x, x \in [0, 1].$$

Find the p.d.f. of $Y = \max(B, C)$ and $Z = B + C$.

12. Let S_n be a binomial (n, p) random variable. For a constant c fixed, find an upper bound of $\mathbb{P}(|S_n - np| > c)$ by Chebychev inequality.

Solution: Since $\mathbb{E}S_n = np$ and $\mathrm{Var}S_n = np(1 - p)$,

$$\mathbb{P}(|S_n - np| > c) \le \frac{\mathrm{Var}S_n}{c^2} = \frac{np(1 - p)}{c^2}.$$

13. Throw a fair die 100 times, and let S_{100} denote the sum of the 100 numbers.

(a) Find $\mathbb{E}S_{100}$ and $\mathrm{Var}(S_{100})$.

Solution: First find the mean and variance of each number. $\mathbb{E}X_1 = 3.5$ and $\mathrm{Var}X_1 = \dfrac{1}{6}(2.5^2 + 1.5^2 + 0.5^2 + 0.5^2 + 1.5^2 + 2.5^2) = \dfrac{35}{12}$.

Then, $\mathbb{E}S_{100} = 100\mathbb{E}X_1 = 350$. $\mathrm{Var}S_{100} = 100\mathrm{Var}X_1 = \dfrac{875}{3}$.

(b) Approximate $\mathbb{P}(S_{100} > 400)$ by a normal distribution in form of

$$\mathbb{P}(S_{100} > 400) \approx \mathbb{P}(Z > z)$$

where Z is a standard normal distribution. Find z.

Solution:

$$\mathbb{P}(S_{100} > 400) = \mathbb{P}\left(\frac{S_{100} - 350}{\sqrt{875/3}} > \frac{400 - 350}{\sqrt{875/3}}\right).$$

So $z = 50/\sqrt{875/3} \approx 2.93$.

14. A device runs until either of two components fails, at which point the device stops running. The joint density function of the lifetimes of the two components, both measured in hours, is

$$f(x, y) = \frac{x + y}{8}, (x, y) \in [0, 2]^2$$

and $f(x, y) = 0$ elsewhere. What is the probability that the device fails during its first hour of operation?

Solution: Let X be the time that the first component fails, and Y the time that the second fails. Here X and Y are random variables with joint density $f(x, y)$. The probability equals

$$\mathbb{P}(\min(X, Y) \leq 1) = 1 - \mathbb{P}(\min(X, Y) > 1) = 1 - \mathbb{P}(X > 1, Y > 1).$$

Now,

$$
\begin{aligned}
\mathbb{P}(X > 1, Y > 1) &= \int_1^2 \int_1^2 f(x, y) dx dy = \int_1^2 \int_1^2 \frac{x + y}{8} dx dy \\
&= \int_1^2 \int_1^2 \frac{x}{8} dx dy + \int_1^2 \int_1^2 \frac{y}{8} dx dy \\
&= \int_1^2 \int_1^2 \frac{x}{4} dx dy = \int_1^2 \frac{x^2}{8} \Big|_1^2 dy = \frac{3}{8}.
\end{aligned}
$$

Therefore, $\mathbb{P}(\min(X, Y) \geq 1) = \frac{5}{8}$.