Name:

Instructions:

- This is a 55-minute exam. Closed-book. No notes/homework.
- Provide your solutions in a simple and clear manner. In particular, you should be able to read them out aloud without any modification.
- You may want to read all the questions first before starting.

1. Consider random variables $\{X_n\}_{n\in\mathbb{N}}$ defined in a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
   (a) State the definitions of $X_n \to 0$ in probability and $X_n \to 0$ almost surely.
   (b) Suppose in addition that $\{X_n\}_{n\in\mathbb{N}}$ are independent random variables, each with the following distribution:
   $$\mathbb{P}(X_n = 0) = (1-p_n)^2, \mathbb{P}(X_n = 1) = 2p_n(1-p_n), \mathbb{P}(X_n = 2) = p_n^2$$
   for some $p_n \in (0,1)$. (That is, X_n has Binomial distribution with parameter $(2, p_n)$.)
   Provide an appropriate choice of $\{p_n\}_{n\in\mathbb{N}}$ such that as $n \to \infty$, $X_n \to 0$ in probability, but not almost surely, and $X_{n^2} \to 0$ almost surely. Justify your answer.
2. Let $X_1, X_2, \dots$ be i.i.d. random variables with $\mathbb{P}(X_1 = -1) = \mathbb{P}(X_1 = 1) = 1/2$. Consider $Y_n := X_n X_{n+2}$ and $T_n := Y_1 + \cdots + Y_n, n \in \mathbb{N}$.
   Show that
   $$\frac{T_n}{n} \to 0 \text{ in probability.}$$
3. Let $\{X_n\}_{n\in\mathbb{N}}$ be independent random variables with
   $$\mathbb{P}(X_n = 1) = \mathbb{P}(X_n = -1) = \frac{1}{2}(1 - 2^{-n}), \mathbb{P}(X_n = 2^n) = 2^{-n}, n \in \mathbb{N}.$$
   Set $S_n := X_1 + \cdots + X_n$.
   (a) Consider $Y_n := X_n \mathbf{1}_{\{|X_n|\le 1\}}$ and $T_n := Y_1 + \cdots + Y_n, n \in \mathbb{N}$. Show that $S_n - T_n$ is a finite random variable almost surely.
   (b) Prove that
   $$\frac{S_n}{n} \to 0 \text{ a.s.}$$
   You may use the statement in part (a) directly.