Name:

**Instructions:**

- This is a 55-minute exam. Close-book. No notes/homework.
- Provide your solutions in a simple and clear manner. In particular, you should be able to read them out aloud without any modification.

1. Let $X$ be a random variable on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with

$$\mathbb{P}(X > x) = e^{-x}, x > 0.$$

Let $Y$ be another random variable in the same probability space, defined as $Y(\omega) := \min(X(\omega), 0.2)$. Let $\mu_Y$ denote the distribution of $Y$. In this example, there exists a unique decomposition

$$\mu_Y = \mu_C + \mu_D$$

with continuous and discrete measures $\mu_C$ and $\mu_D$, respectively. Describe $\mu_C$ and $\mu_D$. **You are not required to provide all the steps for this problem.**

2. Let $X, Y$ be non-negative random variables defined on a common probability space. The two random variables are *not necessarily independent.* Show that

$$\mathbb{E}\left(XY^2\right) = 2\int_0^\infty \int_0^\infty y\mathbb{P}(X > x, Y > y)dxdy.$$

Hint: Fubini's theorem.

3. Let $\{X_n\}_{n\in\mathbb{N}}$ be a collection of random variables on a common probability space, such that $X_n \uparrow X$ as $n \to \infty$ almost surely, and $\mathbb{E}X_1^- < \infty$. Show that $\mathbb{E}X_n \uparrow \mathbb{E}X$ as $n \to \infty$.

Mark clearly where you use the assumption $\mathbb{E}X_1^- < \infty$. When using a limit theorem, state and verify all the conditions.

Hint: consider $Y_n := X_n + X_1^-$.