Name:

**Instructions:**

- This is a two-hour exam. Closed-book. No notes/homework.
- Provide your solutions in a simple and clear manner. In particular, you should be able to read them out aloud without any modification.
- When using main theorems/results from the textbook and/or homework, state and verify the conditions carefully.

1. Suppose $\{X_n\}_{n\in\mathbb{N}}$ and $X$ are random variables such that $\lim_{n\to\infty} X_n = X$ almost surely.

   (a) State the extra condition in the dominated convergence theorem such that
   $$\lim_{n\to\infty} \mathbb{E}X_n = \mathbb{E}X. \tag{1}$$

   (b) By providing a counterexample, show that without the extra condition in part (a), (1) does not necessarily hold.

2. Let $\{X_n\}_{n\in\mathbb{N}}$ be i.i.d. Bernoulli random variables with parameter $p \in (0,1)$: $\mathbb{P}(X_1 = 1) = p = 1 - \mathbb{P}(X_1 = 0)$. Consider
   $$S_n := \sum_{k=1}^{n} kX_k, n \in \mathbb{N}.$$

   (a) Compute $\mathbb{E}S_n$ and $\mathrm{Var}(S_n)$.

   (b) Find a sequence $\{a_n\}_{n\in\mathbb{N}}$ of real numbers such that
   $$\frac{S_n}{a_n} \to 1 \text{ in probability.}$$
   Justify your answer.

   You might need the following formulas:
   $$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \quad \text{and} \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, n \in \mathbb{N}.$$

3. Consider the following version of the renewal problem: let $\{X_n\}_{n\in\mathbb{N}}$ be i.i.d. non-negative random variables with $\mathbb{E}X_1 \in (0,\infty)$. Consider $S_n := X_1 + \cdots + X_n$ and

$$N_t := \max\{n \in \mathbb{N} : S_n \le t\}, t \ge 0,$$

with the convention that $\max\emptyset \equiv 0$.

   (a) State the strong law of large numbers.

   (b) Prove
$$\lim_{t\to\infty} \frac{N_t}{t} = \frac{1}{\mathbb{E}X_1} \text{ a.s.}$$
   Hint: observe $S_{N_t} \le t < S_{N_t+1}$.

4. Let $\{X_n\}_{n\in\mathbb{N}}$ be a collection of independent random variables with

$$\mathbb{P}(X_n = \pm n) = \frac{1}{n^\beta} \quad \text{and} \quad \mathbb{P}(X_n = 0) = 1 - \frac{1}{n^\beta}, n \in \mathbb{N},$$

where $\beta \in (0,1)$ is fixed for all $n \in \mathbb{N}$. Consider $S_n := X_1 + \cdots + X_n$.

   (a) Show that
$$\mathrm{Var}(S_n) \sim Cn^{3-\beta}, \tag{2}$$
   and identify the constant $C$ as a function of $\beta$.
   You may use the formula
$$\sum_{k=1}^{n} k^\theta \sim \frac{n^{\theta+1}}{\theta+1}$$
   for $\theta > 0$. Recall that by $a_n \sim b_n$ we mean $\lim_{n\to\infty} a_n/b_n = 1$.

   (b) Show that
$$\frac{S_n}{n^\gamma} \Rightarrow \mathcal{N}(0,\sigma^2)$$
   for some $\sigma > 0, \gamma > 0$. Identify $\sigma$ and $\gamma$ as functions of $\beta$. (Or, to get partial credit, express them in terms of $C$ in (2) and $\beta$.)