Due Fri. Apr. 14 in class.

1. Read lecture notes Chapter 4.2–4.3.

2. Suppose $\{X_n\}_{n\in\mathbb{N}}$ are independent random variables with
$$\mathbb{P}(X_n = \pm n) = \frac{1}{2n^2}, \mathbb{P}(X_n = \pm 1) = \frac{1}{2}\left(1 - \frac{1}{n^2}\right), n \in \mathbb{N}.$$
   (a) Show that $\lim_{n\to\infty} \mathrm{Var}(S_n)/n = 2$.
   (b) Show that $S_n/\sqrt{n} \Rightarrow \mathcal{N}(0,1)$. Note that none of the two central limit theorems can be applied directly.
   Hint: Occasionally X_n may take big values $\pm n$: the magnitudes of these values cause the conditions of the central limit theorem to fail, but fortunately such big values disappear eventually (easily seen from first Borel–Cantelli lemma) and hence should not have any impact in the limit (since the normalization goes to infinite). The idea is again to treat the big (and rare) values separately, by considering $Y_n := X_n \mathbf{1}_{\{|X_n|\le 1\}}$ and $T_n := Y_1 + \cdots + Y_n$.

   Comment: this is an example to keep in mind that $Z_n \Rightarrow Z$ does not necessarily imply that $\lim_{n\to\infty} \mathbb{E}|Z_n|^\alpha = \mathbb{E}|Z|^\alpha$ ($\alpha = 2$ in this case).

3. (Prelim April 2015, 5) Let $\{U_n\}_{n\in\mathbb{N}}$ be a collection of i.i.d. random variables distributed uniformly on interval $(0,1)$. Consider a triangular array of random variables $\{X_{n,k}\}_{k=1,\dots,n,n\in\mathbb{N}}$ defined as
$$X_{n,k} = \mathbf{1}_{\{\sqrt{n}U_k \le 1\}} - \frac{1}{\sqrt{n}}.$$
   Find constants $\{a_n, b_n\}_{n\in\mathbb{N}}$ such that
$$\frac{X_{n,1} + \cdots + X_{n,n} - b_n}{a_n} \Rightarrow \mathcal{N}(0,1) \text{ as } n \to \infty.$$

4. (Prelim August 2015, 5) Let $\{U_n\}_{n\in\mathbb{N}}$ be a collection of i.i.d. random variables with $\mathbb{E}U_n = 0$ and $\mathbb{E}U_n^2 = \sigma^2 \in (0,\infty)$. Consider random variables $\{X_n\}_{n\in\mathbb{N}}$ defined by $X_n := U_n + U_{2n}, n \in \mathbb{N}$, and the partial sum $S_n = X_1 + \cdots + X_n$. Find constants $\{a_n, b_n\}_{n\in\mathbb{N}}$ such that
$$\frac{S_n - b_n}{a_n} \Rightarrow \mathcal{N}(0,1) \text{ as } n \to \infty.$$

5. (Prelim April 2016, 8) Use Lindeberg–Feller central limit theorem to prove the following. Given a sequence $\{X_n\}_{n\in\mathbb{N}}$ of independent but *not necessarily identically distributed* random variables with mean zero, variance 2 and $\mathbb{E}|X_k|^4 < 17$, show that the central limit theorem for the partial sums $S_n := X_1 + \cdots + X_n$ holds in the form of
$$\frac{S_n - b_n}{a_n} \Rightarrow \mathcal{N}(0,1) \text{ as } n \to \infty,$$
   and specify the normalizing constants $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$.

6. List all the previous prelim problems (number and year, in probability), if any, that you would like to see a discussion in class.