Due Fri. Apr. 7 in class.

1. Read lecture notes Chapter 4.1.

2. Let $\{X_n\}_{n\in\mathbb{N}}$ be i.i.d. standard exponential random variables. That is, $\mathbb{P}(X_1 > x) = e^{-x}, x > 0$. Show that
$$\max_{i=1,\dots,n} X_i - \log n$$
converges weakly as $n \to \infty$ to the so-called Gumbel distribution, which has cumulative distribution function $F(x) = \exp(-e^{-x}), x \in \mathbb{R}$.

   Hint: prove by definition.

3. (Prelim April 2014 (4a)) Let $\{X_n\}_{n\in\mathbb{N}}$ be i.i.d. random variables distributed uniformly in $(0,\theta)$, for some $\theta > 0$. That is, $\mathbb{P}(X_1 \le x) = x/\theta, x \in (0,\theta)$. Set $M_n := \max_{i=1,\dots,n} X_i$. Prove that
$$Z_n := n\left(\theta - \frac{n+1}{n}M_n\right)$$
converges weakly to a non-degenerate distribution, as $n \to \infty$. Identify the limit distribution.

   Hint: prove by definition.

4. (Prelim May 2015, 3(a)) Let $\{X_n\}_{n\in\mathbb{N}}$ be i.i.d. random variables with probability density function, for some fixed $\theta > 0$,
$$p(x) = \theta(\theta+1)x^{\theta-1}(1-x), 0 < x < 1.$$
Consider
$$T_n := \frac{2S_n/n}{1 - S_n/n}, n \in \mathbb{N}.$$
Show that
$$\sqrt{n}(T_n - \theta) \Rightarrow \mathcal{N}(0, \sigma^2(\theta)),$$
and identify the expression of $\sigma^2(\theta)$.

   Hint: first verify $\mathbb{E}X_1 = \theta/(\theta+2)$. Then $\sqrt{n}(T_n - \theta)$ can be rewritten in the form of
$$\sqrt{n}(T_n - \theta) = \frac{S_n - \mathbb{E}S_n}{\sqrt{n}} \cdot Z_n$$
for some random variable Z_n.

5. **(Problem for fun and extra credit)** Simulate a sequence of i.i.d. random variables $\{X_n\}_{n\in\mathbb{N}}$, and plot the partial sums $S_1, \dots, S_n$ as a (random) line. Start with $\mathbb{P}(X_i = \pm 1) = 1/2$.

   (a) How does the plot change as n becomes larger and larger?

   (b) What about taking other distributions for the steps $\{X_n\}_{n\in\mathbb{N}}$? Say, Gaussian, exponential, uniform, etc...

   (c) What happens if each X_i is a Cauchy random variable?

   Comment: the sequence $\{S_n\}_{n\in\mathbb{N}}$ is known as a *random walk*.