Due Wed. Mar. 22 in class.

1. Read lecture notes Chapter 3.6.

2. Let $X_1, X_2, \ldots$ be i.i.d. random variables with exponential distribution: $\mathbb{P}(X_1 > x) = e^{-x}, x > 0$. We prove

$$\limsup_{n\to\infty} \frac{X_n}{\log n} = 1 \text{ a.s.}$$

For this purpose, we proceed in two steps.

(a) Show

$$\mathbb{P}\left(\frac{X_n}{\log n} > 1 + \epsilon \text{ i.o.}\right) = 0, \text{ for all } \epsilon > 0.$$

(b) Show

$$\mathbb{P}\left(\frac{X_n}{\log n} \geq 1 - \epsilon \text{ i.o.}\right) = 1, \text{ for all } \epsilon > 0.$$

3. Suppose a light bulb in the math department lounge burns for an amount of time X, and then remains burned out for an amount of time Y until being replaced. Let X_i and Y_i denote the corresponding times for the i-th light bulb. Assume that $\mathbb{E}X_1 < \infty$ and $\mathbb{E}X_2 < \infty$. All these random variables are assumed to be independent. Let R_t denote the amount of time during the period $[0, t]$ such that the light bulb is working. Show that

$$\lim_{t\to\infty} \frac{R_t}{t} = \frac{\mathbb{E}X_1}{\mathbb{E}X_1 + \mathbb{E}Y_1} \text{ a.s.}$$

(a) Consider $Z_n := X_n + Y_n, n \in \mathbb{N}, S_n := Z_1 + \cdots + Z_n, S_0 := 0$, and

$$N_t := \sup\{n \in \mathbb{N} \cup \{0\} : S_n \leq t\}, t > 0.$$

Consider $T_n := X_1 + \cdots + X_n, T_0 := 0$. Then one can write

$$R_t = T_{N_t} + \zeta_t \tag{1}$$

for some non-negative random variable ζ_t. Express ζ_t in terms of random variables X, Y, S, T and N.

(b) From (1) it follows that $T_{N_t} \leq R_t < T_{N_t+1}$ almost surely. Prove the desired result. Hint: write

$$\frac{T_{N_t}}{t} = \frac{T_{N_t}}{N_t} \cdot \frac{N_t}{t}.$$

4. Mr. Smith decided to investigate a total wealth of $W_0 = w > 0$ (in dollars) from next year. By the end of n-th year, his investment becomes W_n, and he reinvestigates all W_n at the beginning of the next year, using the same strategy. His strategy at the beginning of each year is the following: (a) a total $p \cdot 100\%$ of his wealth is spent to buy bonds, which yields \$$a$ for each \$1 investigated by the end of the year;

(b) the rest $(1-p)\cdot 100\%$ is spent to buy stocks, which yields V_n for each \$1 investigated by the end of the year. In short, we have

$$W_n := (ap + (1-p)V_n)W_{n-1}, n \in \mathbb{N}.$$

Assume $a > 0$, $p \in [0,1]$ and $\{V_n\}_{n\in\mathbb{N}}$ are i.i.d. non-negative random variables.

(a) Show that $\lim_{n\to\infty} n^{-1} \log W_n = c$ almost surely for some constant c. Provide an expression of c.

(b) Suppose $\mathbb{P}(V_1 = 1) = \mathbb{P}(V_1 = 4) = 1/2$. The c depends only on a and p. Determine the optimal investment strategy p as a function of a, so that c is maximized.

5. $(***)$ [Numerical justification of limit theorems]

Justify, by numerical simulations, limit theorems in the following examples that we have seen in class: the coupons collector problem (Example 3.3.2), box occupation problem (Example 3.3.4), and St. Petersburg's parabox (Example 3.4.3).

Pick your most favorite example from above, and write an R code (or by your favorite programming language) to run numerical simulations that could provide justifications to theoretical results. Is your result convincing (and as expected)? Provide a brief summary of your experiments, using either plots or tables, and in particular explain what is your *design of the simulation* and *how the outcomes support the limit theorem.*

Hint: you may want to start by thinking of *how to justify/demonstrate the Strong Law of Large Numbers by simulations.* The simulations will not be as convincing as proofs, but they can be very helpful when explaining limit theorems to people not having enough background in (proofs of) probability theory.

This problem is for extra credits only. Deadline before the exam week.