Due Wed. Mar. 8 in class.

1. Read lecture notes 3.3 – 3.5.

2. Let $X_1, X_2, \dots$ be i.i.d. random variables with distribution determined by $\mathbb{P}(X_1 = (-1)^k k) = C^{-1}(k^2 \log k)^{-1}, k \geq 2$, with $C = \sum_{k=2}^{\infty}(k^2 \log k)^{-1}$.

   (a) Show $\mathbb{E}|X_1| = \infty$.

   (b) Find a sequence μ_n such that $S_n/n - \mu_n \to 0$ in probability.

   (c) Find a value μ such that $\lim_{n\to\infty} \mu_n = \mu$ and conclude that

   $$\lim_{n\to\infty} S_n/n = \mu \text{ in probability.}$$

   You may use the fact $\lim_{n\to\infty} \sum_{k=1}^{n}(-1)^k(k \log k)^{-1}$ exists and is finite.

3. Let $\{A_n\}_{n\in\mathbb{N}}$ be a collection of events, and consider the sets $\{A_n \text{ i.o.}\}$, $\{A_n^c \text{ i.o.}\}$, $\{A_n \text{ eventually}\}$ and $\{A_n^c \text{ eventually}\}$. Which pair of sets are complements of each other? Find all the pairs.

4. Given any sequence $\{p_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} p_n = 0$, provide a concrete example of a sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ defined on an appropriate probability space, such that each X_n is a Bernoulli(p_n) random variable, and $X_n \to 0$ almost surely.

5. Let $X_1, X_2, \dots$ be i.i.d. random variables, $m \in \mathbb{N}$ and $f : \mathbb{R}^m \to \mathbb{R}$ a bounded measurable function. Consider $Y_n := f(X_n, \dots, X_{n+m-1})$ and $T_n := Y_1 + \cdots + Y_n$.

   (a) Random variables $\{Y_n\}_{n\in\mathbb{N}}$ have the following property: there exists some $\ell \in \mathbb{N}$, such that for all $i \neq j, |i - j| > \ell$, Y_i and Y_j are independent. In such a case, $\{Y_n\}_{n\in\mathbb{N}}$ are referred to as *ℓ-dependent*. What is the smallest value ℓ in this example?

   (b) Show that

   $$\lim_{n\to\infty} \frac{T_n}{n} = \mu \text{ in probability}$$

   for some $\mu \in \mathbb{R}$. Express μ.

   Hint: Use the fact that $\{Y_n\}_{n\in\mathbb{N}}$ is a stationary sequence of random variables, and investigate the covariance function φ. Show that $\varphi : \mathbb{N} \to \mathbb{R}$ is a bounded function. For what values of i do we have $\varphi(i) = 0$?

   (c) Prove that

   $$\lim_{n\to\infty} \frac{T_n}{n} = \mu \text{ a.s.}$$

   Hint: consider $T_n^{(\ell)} = Y_\ell + Y_{m+\ell} + \cdots + Y_{(n-1)m+\ell}, \ell = 1, \dots, m$. What can you say about $T_n^{(\ell)}/n$ as $n \to \infty$?

Comments: here, part (c) is a strictly stronger result than part (b). However, the method in part (c) depends crucially on the ℓ-dependence assumption, while the method in part (b) is more general.