Due Wed. Feb 29 in class. Problems with $(**)$ are for extra credits only.

1. Read lecture notes 3.1 – 3.2.

2. Recall the definition of convergence in probability: we say X_n converges to X, if
$$\lim_{n\to\infty} \mathbb{P}(|X_n - X| > \epsilon) = 0, \text{ for all } \epsilon > 0. \tag{1}$$
Consider another condition:
$$\lim_{n\to\infty} \mathbb{P}(X_n \neq X) = 0. \tag{2}$$
Show that (2) implies (1), but the opposite is not necessarily true without further assumption.

3. Let $\{X_n\}_{n\in\mathbb{N}}$ be a stationary sequence of random variables with covariance function φ, so that $\mathrm{Cov}(X_i, X_j) = \varphi(i-j)$. Write $S_n = X_1 + \cdots + X_n$. Show that
$$\mathrm{Var}(S_n) = \sum_{i=-n+1}^{n-1} (n - |i|)\varphi(i) = n\varphi(0) + 2\sum_{i=1}^{n-1}(n-i)\varphi(i).$$

4. Let $X_1, X_2, \ldots$ be uncorrelated random variables such that $\mathbb{E}X_i = \mu_i$ and $\lim_{i\to\infty} \mathrm{Var}(X_i)/i = 0$. Consider $S_n = X_1 + \cdots + X_n$, and $\nu_n = \mathbb{E}S_n/n$. Show that $S_n/n - \nu_n \to 0$ in L^2 and in probability.

   Hint: Compute $\mathrm{Var}(S_n)$ and find an upper bound for $\mathrm{Var}(S_n)/n^2$. You may use the fact that for any sequence of real numbers $\{a_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} a_n = a \in \mathbb{R}$,
$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} a_i = a.$$

5. (2016 April Prelim) Consider i.i.d. Poisson random variables $\{X_n\}_{n\in\mathbb{N}}$ with parameter $\lambda > 0$. Consider $Y_n := X_n X_{2n}, n \in \mathbb{N}$ and $T_n := Y_1 + \cdots + Y_n$.

   (a) Compute $\mathbb{E}T_n$.

   (b) Find an explicit constant C such that $\mathrm{Var}(T_n) \leq Cn$ for all $n \in \mathbb{N}$. Explain clearly how you determine the constant C. The constant does not have to be optimal and may depend on λ. However, it must be independent from n.

   Hint: $\{Y_n\}_{n\in\mathbb{N}}$ *is not a stationary sequence, so the calculation of* $\mathrm{Var}(S_n)$ *before cannot be used.* However, the calculation of $\mathrm{Var}(T_n)$ is not much more complicated, and here we need only an upper bound instead of an exact expression.

   (c) Find a sequence of real numbers $\{a_n\}_{n\in\mathbb{N}}$, such that
$$\frac{T_n}{a_n} \to 1 \text{ in probability.}$$
   Justify your result.