Due Wed. Feb 22 in class. Problems with $(**)$ are for extra credits only.

1. Read lecture notes 2.4–2.5.

2. (2014 August Prelim) Let X and Y be two non-negative random variables defined on a common probability space. Show that

$$\mathbb{E}\left(\frac{1}{XY}\right) = \int_0^\infty \int_0^\infty \frac{\mathbb{P}(X \le x, Y \le y)}{x^2 y^2} dx dy.$$

   Hint: Fubini's theorem.

3. Prove the following properties of distributions by using the formula for the distribution of sum of independent random variables, and identify the parameter of the distribution of the sum and minimum respectively in each case.

   (i) The sum of n independent Poisson random variables with parameters $\lambda_1, \ldots, \lambda_n$ respectively is still Poisson.

   (ii) $(**)$ The sum of n independent Gaussian random variables with parameters $(\mu_i, \sigma_i^2), i = 1, \ldots, n$ respectively is still Gaussian.

   (iii) The minimum of independent exponential random variables with parameter $\lambda_1, \ldots, \lambda_n$ respectively is still exponential.

   Note that it suffices to prove for $n = 2$, and conclude by induction.

4. Consider 3 i.i.d. random variables $\{X_i\}_{i=1,2,3}$ such that $\mathbb{P}(X_i = \pm 1) = 1/2$ for $i = 1, 2, 3$. Consider $Y_1 = X_1 X_2$, $Y_2 = X_2 X_3$, and $Y_3 = X_3 X_1$. Are $\{Y_i\}_{i=1,2,3}$ pairwise independent? Are they independent?

5. (**) Provide an example of 3 events A_1, A_2, A_3, such that

$$\mathbb{P}\left(\bigcap_{i=1}^3 A_i\right) = \prod_{i=1}^3 \mathbb{P}(A_i),$$

   but they are not independent.

6. Provide an example of two random variables X and Y, such that X and Y are not independent, but $\mathbb{E}(XY) = \mathbb{E}X\mathbb{E}Y$. Such random variables are usually referred to as *uncorrelated*. Hint: consider $X = Z_1 Z_2$ and $Y = Z_2 Z_3$.

7. $(*)$ (Review exercise on calculus, hand-in not required.) Exercise 2.4.2 from lecture notes.