Due Fri. February 10 in class. Problems with $(**)$ are for extra credits only.

1. Read lecture notes Sections 2.1 – 2.4.

2. Let $p \in (0,1)$ be arbitrary. Construct 3 Bernoulli random variables X with parameter p, (that is, $\mathbb{P}(X = 1) = p = 1 - \mathbb{P}(X = 0)$) in three different probability spaces:

   (i) $((0,1), \mathcal{B}((0,1)), \mathrm{Leb})$,

   (ii) $(\{0,1\}, 2^{\{0,1\}}, \mathbb{P})$ for some $\mathbb{P}$ to choose,

   (iii) $([0,\infty), \mathcal{B}([0,\infty)), \mathbb{P})$ where $\mathbb{P}$ is determined by $\mathbb{P}((a,b)) = \int_a^b e^{-x}dx, 0 \le a < b < \infty$.

   In each case, express explicitly the random variable as a function on the probability space.

3. Construct two different random variables in a common probability space, so that they have the same Bernoulli distribution with parameter $p \in (0,1)$. Namely, provide explicitly a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and two measurable functions $X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and $Y : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, such that the two random variables have the same Bernoulli distribution with parameter p, and $\mathbb{P}(\omega : X(\omega) \neq Y(\omega)) > 0$.

4. Consider the measurable space $((0,1), \mathcal{B}((0,1)))$, and probability measures $\mathbb{P}_1 := \mathrm{Leb}$ and
$$\mathbb{P}_2(\cdot) := \sum_{n=1}^{\infty} \frac{1}{2^n} \delta_{\frac{1}{n+1}}(\cdot).$$
   Consider $X(\omega) := \omega^2, \omega \in (0,1)$.

   (a) Compute $\mathbb{P}_1(X \ge 1/4)$ and $\mathbb{P}_2(X \ge 1/4)$.

   (b) The random variable X under $\mathbb{P}_1$ and $\mathbb{P}_2$ has different distributions respectively, denoted by say μ_1 and μ_2. Express μ_1 and μ_2.

   (c) $(**)$ X is a random variable in $((0,1), \mathcal{B}((0,1)), \mathbb{P}_1)$ and $((0,1), \mathcal{B}((0,1)), \mathbb{P}_2)$, respectively. Express its cumulative distribution function $F_i(x) = \mathbb{P}_i(X \le x)$ in each case.

5. Let X be a random variable with $\mathbb{E}|X| < \infty$. Show that
$$\lim_{n\to\infty} \mathbb{E}(X\mathbf{1}_{\{|X|\le n\}}) = \mathbb{E}X \quad \text{and} \quad \lim_{n\to\infty} \mathbb{E}(X\mathbf{1}_{\{|X|>n\}}) = 0.$$
   If instead of $\mathbb{E}|X| < \infty$ assume that X is non-negative, then do the above conclusions still hold? Justify your answer.

6. (**) Consider a non-negative random variable X. Show that
$$\lim_{y\to\infty} y\mathbb{E}\left(\frac{1}{X}\mathbf{1}_{\{X>y\}}\right) = 0.$$

7. (**) Let X and Y be two non-negative random variables defined on a common probability space. Show that
$$\mathbb{E}\left(\frac{1}{XY}\right) = \int_0^\infty \int_0^\infty \frac{\mathbb{P}(X \le x, Y \le y)}{x^2y^2} dxdy.$$
   Hint: Fubini's theorem.

8. Let X be a random variable with uniform $(0,1)$ distribution ($\mathbb{P}(X \le x) = x, x \in (0,1)$), defined on certain probability space. Consider random variable $Y(\omega) := \max\{X(\omega), 1/3\}$.

   (i) Describe the distribution μ_Y of Y in the form of
$$\mu_Y = \mu_C + \mu_D$$
   where μ_1 and μ_2 are discrete and continuous measures, respectively.

   (ii) Compute $\mathbb{E}(Y^2)$.