Due Fri. Feburary 3 in class. Problems with $(**)$ are for extra credits only.

1. Read lecture notes 1.3 – 1.5.

2. Consider
$$F(x) = \begin{cases} 0 & x < 0 \\ 0.2 + x & 0 \le x < 0.2 \\ 0.4 + x & 0.2 \le x < 0.5 \\ 1 & x \ge 0.5 \end{cases}$$
Let μ denote the measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ determined by F. Compute $\int x d\mu(x), \int x^2 d\mu(x)$.

3. For each of the conditions below, provide an example of a sequence of measurable functions $\{f_n\}_{n\in\mathbb{N}}$ defined on an appropriate measure space $(\Omega, \mathcal{F}, \mu)$ that satisfies.
   (i) $f_n \uparrow f$ as $n \to \infty$, and $\lim_{n\to\infty} \int f_n d\mu \ne \int f d\mu$.
   Hint: compare to the monotone convergence theorem.
   (ii) $\mu(\Omega) = 1$, $\lim_{n\to\infty} f_n = f$ and $\lim_{n\to\infty} \int f_n d\mu \ne \int f d\mu$.
   Hint: compare to the dominated convergence theorem.

4. Let $\{b_{m,n}\}_{m,n\in\mathbb{N}}$ be a collection of non-negative real numbers. Consider
$$S_n := \sum_{m\in\mathbb{N}} b_{m,n}, n \in \mathbb{N},$$
and its limit. Assume that for each m, $\lim_{n\to\infty} b_{m,n} = \beta_m \in \mathbb{R}$ and $\sum_{m=1}^{\infty} \beta_m < \infty$.
   (i) What can one say about $\liminf_{n\to\infty} S_n$ and $\sum_{m=1}^{\infty} \beta_m$ by Fatou's lemma?
   Hint: recall that we now view infinite series as Lebesgue integrals; consider $(\Omega, \mathcal{F}) = (\mathbb{N}, 2^{\mathbb{N}})$ equipped with measure $\mu = \sum_{m=1}^{\infty} \delta_m$, and $f_n(m) := b_{m,n}, m, n \in \mathbb{N}$.
   (ii) By providing a counterexample, show that one should not expect $\lim_{n\to\infty} S_n$ to exist as a finite number, without further assumptions on $b_{m,n}$.
   (iii) Provide an additional assumption on $b_{m,n}$ that guarantees
$$\lim_{n\to\infty} S_n = \sum_{m=1}^{\infty} \beta_m.$$
   Hint: as in part (i), formulate the problem in the framework of the dominated convergence theorem.

5. Let f be a measurable function on $(\Omega, \mathcal{F}, \mu)$ with $\mu(\Omega) < \infty$, and $p > 0$. Prove that if $\int |f|^p d\mu < \infty$, then for all $q \in (0, p)$, $\int |f|^q d\mu < \infty$. Hint: write $f = f \cdot 1$, and use Hölder's inequality.

6. $(**)$ Find an example of functions $\{f_n\}_{n\in\mathbb{N}}$ on $((0,1), \mathcal{B}((0,1)), \lambda \equiv \text{Leb})$, such that
$$\int \left(\sum_{n=1}^{\infty} f_n \right) d\lambda \ne \sum_{n=1}^{\infty} \int f_n d\lambda.$$

7. Let μ be a finite measure on $\mathbb{R}$ and $F(x) = \mu((-\infty, x])$. Show that
$$\int F(x + c) - F(x) dx = c\mu(\mathbb{R}).$$
Hint: write the integrand as another integral, and apply Fubini's theorem.