Due Fri. January 20 in class. Problems with $(**)$ are for extra credits only.

1. Read lecture notes Sections 1.1, 1.2.

2. Let $\Omega = \mathbb{N}$, and $\mathcal{F}$ be the collection of all subsets A of $\mathbb{N}$ such that either A or A^c is finite. For all $A \in \mathcal{F}$, set $\mu(A) = 0$ if A is finite, and $\mu(A) = 1$ if A^c is finite.

   (i) Show that $\mathcal{F}$ is *not* a σ-algebra.

   (ii) (**) Does there exist a measure space $(\mathbb{N}, \mathcal{G}, \mu^*)$ such that: $\mathcal{F} \subset \mathcal{G}$, and $\mu^*(A) = \mu(A)$ for all $A \in \mathcal{F}$?

3. Provide an example of a measure space $(\Omega, \mathcal{F}, \mu)$ with a family of measurable sets $\{A_n\}_{n\in\mathbb{N}}$ such that $A_n \downarrow A \in \mathcal{F}$ as $n \to \infty$ but $\lim_{n\to\infty} \mu(A_n) \neq \mu(A)$.

4. Consider the measurable space $((0,1), \mathcal{B}((0,1)))$ and subsets $A_n = [2^{-2n}, 2^{-2n+1}), n \in \mathbb{N} := \{1, 2, \dots\}$. Let μ_F denote the measure determined by the Stieltjes measure function F below. Compute $\mu_F(\bigcup_{n=1}^{\infty} A_n)$ in each case.

   (i) $F(x) = x^2$.

   (ii) $F(x) = \mathbf{1}_{(-\infty,1/4)}(x) + 6x\mathbf{1}_{[1/4,\infty)}(x)$.

5. Consider $\Omega = \{1, 2, 3, 4\}$.

   (i) Find a strictly increasing sequence of σ-algebras $\mathcal{F}_1 \subsetneq \mathcal{F}_2 \subsetneq \cdots \subsetneq \mathcal{F}_n$ ($\mathcal{A} \subsetneq \mathcal{B}$ means $\mathcal{A} \subset \mathcal{B}$ and $\mathcal{A} \neq \mathcal{B}$) of Ω. What is the largest n that you can get?

   (ii) Consider $\mathcal{F} = \sigma(\{\{1,2\},\{3,4\}\})$ and $\mathcal{G} = \sigma(\{\{1\},\{2\},\{3,4\}\})$. Provide an example of a function $f : \Omega \to \mathbb{R}$ such that f is measurable with respect to $\mathcal{G}$, but not with respect to $\mathcal{F}$.

6. Consider the following functions

$$f_1(x) = 0, f_2(x) = \mathbf{1}_{\{1/2\}}(x), f_3(x) = \mathbf{1}_{\mathbb{Q}\cap(0,1)}(x),$$

   on the measurable space $((0,1), \mathcal{B}((0,1)))$. We examine these functions under the following *different* measures respectively,

$$\mu_1 = \text{Leb}, \mu_2 = \delta_{1/2}, \mu_3 = \frac{1}{2}\delta_{1/2}, \mu_4 = \sum_{q\in\mathbb{Q}\cap(0,1)} \delta_q, \mu_5 = \mu_1 + \mu_2.$$

   (i) Recall that f *and* g *belong to the same* μ*-equivalent class*, if $f = g$ μ-a.e. For each μ_i above, f_1, f_2 and f_3 belong to how many different equivalent classes? **No need to justify your answer.**

   (ii) Compute

$$\mu(\{f = 1\}) \equiv \mu\left(\{\omega \in (0,1) : f(\omega) = 1\}\right)$$

   for each $\mu_i, i = 4, 5$ and $f_j, j = 1, 2, 3$.

7. $(**)$ Exercise 1.1.17 from Lecture notes.