Maximal Moments and Uniform Modulus of Continuity for Stable Random Fields

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Outline

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1. (Symmetric) Stable random fields

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a real-valued symmetric α-stable random field with the following representation

$$\left\{X(t),\ t \in \mathbb{R}^N\right\} \stackrel{d}{=} \left\{\int_F f(t,x)\, M_\alpha(dx),\ t \in \mathbb{R}^N\right\}, \qquad (1)$$

where M_α is a SαS random measure on a measurable space $(F, \mathcal{F})$ with control measure m, $f(t,\cdot) : F \to \mathbb{R}$ $(t \in \mathbb{R}^N)$ is a family of functions on F satisfying

$$\int_F |f(t,x)|^\alpha\, m(dx) < \infty, \qquad \forall\, t \in \mathbb{R}^N. \qquad (2)$$

For any $n \geq 1$ and $t^1, \ldots, t^n \in \mathbb{R}^N$, the characteristic function of $X(t^1), \ldots, X(t^n)$ is given by

$$\mathbb{E}\exp\left(i\sum_{j=1}^{n} \xi_j X(t^j)\right) = \exp\left(-\Big\|\sum_{j=1}^{n} \xi_j f(t^j, \cdot)\Big\|_{\alpha,m}^{\alpha}\right),$$

where $\xi_j \in \mathbb{R}$ $(1 \leq j \leq n)$ and $\|\cdot\|_{\alpha,m}$ is the $L^\alpha(F, \mathcal{F}, m)$-norm if $\alpha \geq 1$ and quasi norm if $0 < \alpha < 1$.
See Samorodnitsky and Taqqu (1994) for more information.

We will also consider real-valued stable random fields X represented by

$$\left\{X(t),\, t \in \mathbb{R}^N\right\} \stackrel{d}{=} \left\{\Re e \int_F f(t,x)\, \widetilde{M}_\alpha(dx),\, t \in \mathbb{R}^N\right\},$$

where $\widetilde{M}_\alpha$ is a complex-valued, rotationally invariant α-stable random measure on a measurable space $(F, \mathcal{F})$ with control measure m and the complex-valued, measurable functions $f(t,\cdot)$ $(t \in \mathbb{R}^N)$ satisfy (2).

2. Some Examples

2.1 Linear fractional stable sheets

Given $0 < \alpha < 2$ and $\vec{H} = (H_1, \ldots, H_N) \in (0,1)^N$, a *linear fractional stable sheet* $Z^{\vec{H}} = \{Z^{\vec{H}}(t), t \in \mathbb{R}^N_+\}$ with values in $\mathbb{R}$ is defined by

$$Z^{\vec{H}}(t) = \int_{\mathbb{R}^N} \prod_{\ell=1}^{N} h_{H_\ell}(t_\ell, s_\ell)\, M_\alpha(ds),$$

where

$$h_{H_\ell}(t_\ell, s_\ell) = \kappa\Big\{ (t_\ell - s_\ell)_+^{H_\ell - \frac{1}{\alpha}} - (-s_\ell)_+^{H_\ell - \frac{1}{\alpha}} \Big\}.$$

Basic properties of $Z^{\vec{H}}$

- Along the j-th direction of $\mathbb{R}_+^N$, $Z^{\vec{H}}$ becomes a real-valued linear fractional stable motion of index H_j.
- $Z^{\vec{H}}$ is operator self-similar in the following sense

$$\left\{Z^{\vec{H}}(At),\ t \in \mathbb{R}_+^N\right\} \stackrel{d}{=} \left\{\Big(\prod_{j=1}^{N} a_j^{H_j}\Big) Z^{\vec{H}}(t),\ t \in \mathbb{R}_+^N\right\}, \tag{3}$$

where $A = (a_{ij})$ is an $N \times N$ diagonal matrix with

$$a_{ij} = \begin{cases} a_i > 0 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

- Modulus of continuity and other properties studied in Ayache, Roueff and Xiao (2009) by using the wavelet method.

2.2 Harmonizable fractional stable sheets

Given $0 < \alpha < 2$ and $\vec{H} = (H_1, \ldots, H_N) \in (0,1)^N$, the *harmonizable fractional stable sheet* $\widetilde{Z}^{\vec{H}} = \{\widetilde{Z}^{\vec{H}}(t), t \in \mathbb{R}^N_+\}$ with values in $\mathbb{R}$ is defined by

$$\widetilde{Z}^{\vec{H}}(t) = \Re e \int_{\mathbb{R}^N} \prod_{j=1}^{N} \frac{e^{it_j\lambda_j} - 1}{|\lambda_j|^{H_j + \frac{1}{\alpha}}} \widetilde{M}_\alpha(d\lambda).$$

Similar to the linear fractional stable sheet, $\widetilde{Z}^{\vec{H}}$ is operator self-similar in the sense of (3). Along each direction of $\mathbb{R}^N_+$, $\widetilde{Z}^{\vec{H}}$ is a real-valued harmonizable fractional stable motion.

3. Uniform Modulus of Continuity

Uniform modulus of continuity of stable random fields have been studied by several authors using different methods.

For example, Kono and Maejima (1991), Takashima (1989), Ayache, Roueff and Xiao (2009), Biermé and Lacaux (2009, 2015), Xiao (2010), Ayache and Boutard (2017), and Panigrahi, Roy, and Xiao (2018).

A modification of the chaining argument

The chaining arguments in the proofs of Kolmogorov's continuity theorem, Dudley's entropy theorem, Talagrand's majorizing measures are powerful for Gaussian and other light-tailed random fields. However, they are not suitable for stable random fields.

X. (2010) modified the chaining argument so that it can be applied to random fields with heavy-tailed distributions.

Let $\{X(t),\ t \in T\}$ be a real-valued random field indexed by a compact metric space (T, ρ).

Let $\{D_n : n \geq 1\}$ be a sequence of finite subsets of T satisfying the following conditions:

(i) There exists an integer κ_0 such that for all $n \geq 1$ and all $\tau_n \in D_n$, we have $\#\big(O_{n-1}(\tau_n)\big) \leq \kappa_0$, where

$$O_{n-1}(\tau_n) := \{\tau'_{n-1} \in D_{n-1} : \rho(\tau_n, \tau'_{n-1}) \leq 2^{-n}\}.$$

(ii) For every $s, t \in T$ with $\rho(s,t) \leq 2^{-n}$, there exist $\{\tau_p(s) : p \geq n\}$ and $\{\tau_p(t) : p \geq n\}$ such that $\tau_n(s) = \tau_n(t)$ and, for every $p \geq n$, $\tau_p(s), \tau_p(t) \in D_p$, $\rho(\tau_p(s), s) \leq 2^{-p}$, $\rho(\tau_p(t), t) \leq 2^{-p}$, and $\tau_p(s) \in O_p(\tau_{p+1}(s))$, $\tau_p(t) \in O_p(\tau_{p+1}(t))$.

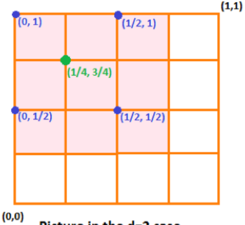
If $s \in D := \cup_{k=1}^{\infty} D_k$, then there exists an integer $q \geq 1$ such that $\tau_p(s) = s$ for all $p \geq q$.

If $T = [0,1]^N$, a natural choice of $\{D_n : n \ge 1\}$ is

$$D_n = \left\{ \left(k_1 2^{-n}, \ldots, k_N 2^{-n}\right) : 0 \le k_j \le 2^n, 1 \le j \le N \right\}.$$

Then $D_n \subset D_{n+1}$ and $\overline{\bigcup_{n=1}^{\infty} D_n} = [0,1]^N$.

The following picture gives D_1 and D_2.



Picture in the d=2 case

Theorem 3.1 [X. (2010)]

Let $\{X(t),\ t \in T\}$ be a real-valued random field indexed by a compact metric space (T, ρ) and let $\{D_n : n \ge 1\}$ be a chaining sequence satisfying (i) and (ii). Suppose $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing continuous function which is regularly varying at the origin with index $\delta > 0$.
If there are constants $\gamma > 0$, and $K > 0$ such that

$$\mathbb{E}\Big(\max_{\tau_n \in D_n} \max_{\tau'_{n-1} \in O_{n-1}(\tau_n)} |X(\tau_n) - X(\tau'_{n-1})|^{\gamma} \Big) \le K \left(\sigma(2^{-n})\right)^{\gamma} \tag{4}$$

for all integers $n \ge 1$, then for all $\epsilon > 0$,

$$\lim_{h \to 0+} \frac{\sup_{t \in T} \sup_{\rho(s,t) \le h} |X(t) - X(s)|}{\sigma(h)(\log 1/h)^{(1+\epsilon)/\gamma}} = 0 \quad \text{a.s.} \tag{5}$$

Proof. Given any $s, t \in D = \bigcup_{k=1}^{\infty} D_k$ with $\rho(s,t) \le 2^{-n}$, let $\{\tau_p(s), n \le p \le q\}$ and $\{\tau_p(t), n \le p \le q\}$ be the two approximating chains to s and t given by Condition (ii). The triangle inequality and the fact $\tau_n(s) = \tau_n(t)$ imply

$$\begin{aligned} &\big|X(t) - X(s)\big| \\ &\le \sum_{p=n+1}^{q} \Big(\big|X(\tau_p(t)) - X(\tau_{p-1}(t))\big| + \big|X(\tau_p(s)) - X(\tau_{p-1}(s))\big| \Big) \\ &\le 2 \sum_{p=n+1}^{\infty} \max_{\tau_p \in D_p} \max_{\tau'_{p-1} \in O_{p-1}(\tau_n)} \big|X(\tau_p) - X(\tau'_{p-1})\big|. \end{aligned} \tag{6}$$

It is helpful to note that, for each $p \ge n+1$, the maximum in (6) is taken over at most $\kappa_0 \, \#(D_p)$ instead of $\#(D_{p-1})\#(D_p)$.

For any integer $n \geq 1$, let

$$\Delta_n = \sup_{s,t \in D} \sup_{\rho(s,t) \leq 2^{-n}} \big| X(t) - X(s) \big|.$$

It follows from (6) and (4) that

$$\mathbb{E}\big(\Delta_n^\gamma\big) \leq K \sum_{p=n+1}^{\infty} \mathbb{E}\Big(\max_{\tau_p \in D_p} \max_{\tau'_{p-1} \in O_{p-1}(\tau_p)} \big| X(\tau_p) - X(\tau'_{p-1}) \big|^\gamma \Big)$$
$$\leq K\, \sigma(2^{-n})^\gamma.$$

Hence Markov's inequality gives that for all integers $n \geq 1$ and real numbers $u > 0$

$$\mathbb{P}\big(\Delta_n \geq \sigma(2^{-n})\, u\big) \leq K\, u^{-\gamma}.$$

The rest follows from a Borel-Cantelli argument.

4. Maximal Moment Estimates

Panigrahi, Roy, and X. (2018) obtained sharp estimates on $\mathbb{E}\big(\max_{1\le k\le n}|\xi_k|^\gamma\big)$ for a stationary SαS random field, and applied them to derive uniform modulus of continuity for self-similar SαS random fields with stationary increments.

- The main tools are ergodic theoretic in nature; see Rosínski (1995, 2000).
- It is based on Samorodnitsky (2004), which is on extreme value theory of SαS processes.
- The solution is strongly connected to the effective dimension of the random field; Roy and Samorodnitsky (2008), Chakrabarty and Roy (2013).

Rosínski (1995, 2000) showed that every stationary SαS random field $\mathbf{Y} = \{Y(t),\ t \in \mathbb{Z}^N\}$ can be represented as

$$Y(t) = \int_S c_t(s) \left(\frac{d\mu \circ \phi_t}{d\mu}(s) \right)^{1/\alpha} f \circ \phi_t(s) M_\alpha(ds), \quad t \in \mathbb{Z}^N, \qquad (7)$$

where $f \in L^\alpha(S, \mu)$, and where

- M_α is a SαS random measure on a standard Borel space $(S, \mathcal{S})$ with σ-finite control measure μ,
- $\{\phi_t\}_{t \in \mathbb{Z}^N}$ is a nonsingular $\mathbb{Z}^N$-action on $(S, \mathcal{S}, \mu)$ (i.e., each $\phi_t : S \to S$ is a measurable map, ϕ_0 is the identity map on S, $\phi_{u+v} = \phi_u \circ \phi_v$ for all $u, v \in \mathbb{Z}^N$ and each $\mu \circ \phi_t$ is equivalent to μ), and
- $\{c_t\}_{t \in \mathbb{Z}^N}$ is a measurable cocycle for $\{\phi_t\}$ (i.e., each c_t is a $\{\pm 1\}$-valued measurable map defined on S satisfying $c_{u+v}(s) = c_u(\phi_v(s))c_v(s)$ for all $u, v \in \mathbb{Z}^N$ and for all $s \in S$).

A measurable set $W \subseteq S$ is called a *wandering set* for the nonsingular $\mathbb{Z}^N$-action $\{\phi_t\}_{t\in\mathbb{Z}^N}$ if $\{\phi_t(W) : \ t \in \mathbb{Z}^N\}$ is a pairwise disjoint collection.

The set S can be decomposed into two disjoint and invariant parts as follows:

$$S = \mathcal{C} \cup \mathcal{D},$$

where $\mathcal{D} = \bigcup_{t\in\mathbb{Z}^d} \phi_t(W^*)$ for some wandering set $W^* \subseteq S$, and $\mathcal{C}$ has no wandering subset of positive μ-measure; see Aaronson (1997).

This decomposition is called the *Hopf decomposition*, and the sets $\mathcal{C}$ and $\mathcal{D}$ are called *c*onservative and *d*issipative parts (of $\{\phi_t\}_{t\in\mathbb{Z}^N}$), respectively.

The action and, correspondingly, the random field $\mathbf{Y}$, is called conservative if $S = \mathcal{C}$ and dissipative if $S = \mathcal{D}$.

Samorodnitsky (2004), Roy and Samorodnitsky (2008) proved the following dichotomy:

$$n^{-N/\alpha} \max_{\|t\|_\infty \le n} |Y(t)| \Rightarrow \begin{cases} c_{\mathbf{Y}} Z_\alpha, & \text{if } \mathbf{Y} \text{ is dissipative,} \\ 0, & \text{if } \mathbf{Y} \text{ is conservative} \end{cases}$$

as $n \to \infty$. Here Z_α is a standard Frechét type extreme value random variable with distribution

$$\mathbb{P}(Z_\alpha \le x) = e^{-x^{-\alpha}}, \ \ x > 0,$$

and $c_{\mathbf{Y}}$ is a positive constant depending on $\mathbf{Y}$.

For conservative actions, Roy and Samorodnitsky (2008) showed that the actual rate of growth of

$$M_n = \max_{\|t\|_\infty \le n} |Y(t)|$$

depends on further properties of the action. Let

$$A = \{\phi_t : \; t \in \mathbb{Z}^N\}$$

be a subgroup of the group of invertible nonsingular transformations on (S, μ) and define a group homomorphism, $\Phi : \mathbb{Z}^N \to A$ by $\Phi(t) = \phi_t$ for all $t \in \mathbb{Z}^N$. Let

$$\mathcal{K} = \mathrm{Ker}(\Phi) = \{t \in \mathbb{Z}^N : \phi_t = 1_S\},$$

where 1_S denotes the identity map on S.

Then $\mathcal{K}$ is a free abelian group and by the first isomorphism theorem of groups, we have

$$A \cong \mathbb{Z}^N / \mathcal{K}.$$

Now, by the structure theorem of finitely generated abelian groups (see, for example, Lang (2002)), we get,

$$A = \bar{F} \oplus \bar{G},$$

where $\bar{F}$ is a free abelian group and $\bar{G}$ is a finite group. Assume $rank(\bar{F}) = p \geq 1$ and $|\bar{G}| = l$.
There exists an injective group homomorphism, $\Psi : \bar{F} \to \mathbb{Z}^N$ such that $\Phi \circ \Psi = 1_{\bar{F}}$. Then $F = \Psi(\bar{F})$ is a free subgroup of $\mathbb{Z}^N$ of rank p, which is the effective dimension of the random field.

Roy and Samorodnitsky (2008) showed that, depending on the nature of the action restricted to F, the asymptotic behavior of the partial maxima can be sharpened:

$$n^{-p/\alpha} M_n \Rightarrow \begin{cases} c_{\mathbf{Y}} Z_\alpha & \text{if the restricted } F\text{-action is dissipative,} \\ 0 & \text{if the restricted } F\text{-action is conservative.} \end{cases}$$

In the representation (7), we denote

$$f_t(s) = c_t(s)\left(\frac{d\mu \circ \phi_t}{d\mu}(s)\right)^{1/\alpha} f \circ \phi_t(s). \tag{8}$$

Analogously to Samorodnitsky (2004), Roy and Samorodnitsky (2008), the asymptotic behavior of the moments of $\{M_n\}$ is related to the deterministic sequence:

$$\{b_n\}_{n\geq 1} = \left\{\left(\int_S \max_{\mathbf{0}\leq t\leq (n-1)\mathbf{1}} |f_t(s)|^\alpha \mu(ds)\right)^{1/\alpha}\right\}_{n\geq 1}. \tag{9}$$

Samorodnitsky (2004), Roy and Samorodnitsky (2008) have proved:

$$n^{-d/\alpha} b_n \to \begin{cases} \tilde{c}_{\mathbf{Y}} & \text{if action is dissipative,} \\ 0 & \text{if action is conservative,} \end{cases} \tag{10}$$

where $\tilde{c}_{\mathbf{Y}}$ is a positive constant.

In the conservative case, a refined result on $\{b_n\}$ in terms of the effective dimension was also proved.

Theorem 4.1 [Panigrahi, Roy, and X. (2018)]

Let $\mathbf{Y} = \{Y(t),\ t \in \mathbb{Z}^N\}$ be a stationary SαS random field with $0 < \alpha < 2$ and having integral representation (7).

- If $\mathbf{Y}$ is dissipative, then for $0 < \beta < \alpha$,

$$n^{-N\beta/\alpha}\mathbb{E}\left[M_n^\beta\right] \to C \quad \text{as } n \to \infty, \tag{11}$$

  where $C > 0$ is a finite constant.
- If $\mathbf{Y}$ is conservative, then for $0 < \beta < \alpha$,

$$n^{-N\beta/\alpha}\mathbb{E}\left[M_n^\beta\right] \to 0 \quad \text{as } n \to \infty. \tag{12}$$

Sketch of Proof of (11)

We will make use of the following series representation for $\{Y_k, \mathbf{0} \le k \le (n-1)\mathbf{1}\}$:

$$Y_k \overset{d}{=} b_n C_\alpha^{1/\alpha} \sum_{j=1}^{\infty} \xi_j \Gamma_j^{-1/\alpha} \frac{f_k(U_j^{(n)})}{\max_{m\in[\mathbf{0},(n-1)\mathbf{1}]} |f_m(U_j^{(n)})|},$$

In the above, $\{\Gamma_n\}_{n\ge1}$ is a sequence of arrival times of a unit rate Poisson process on $(0,\infty)$, $\{\xi_n\}_{n\ge1}$ are i.i.d. Rademacher random variables, and $\{U_\ell^{(n)}\}_{n\ge1}$ ($\ell = 1, 2$) are i.i.d. S-valued random variables with common law η_n whose density is given by

$$\frac{d\eta_n}{d\mu} = b_n^{-\alpha} \max_{\mathbf{0}\le t\le(n-1)\mathbf{1}} |f_t(s)|^\alpha, \quad s \in S.$$

All four sequences are independent.

By applying Samorodnitsky (2004), one has the following upper bound on the tail distribution of $b_n^{-1} M_n$:

$$\mathbb{P}\big(b_n^{-1} M_n > \lambda\big) \le \mathbb{P}\big(C_\alpha^{1/\alpha} \Gamma_1^{-1/\alpha} > \lambda(1-\delta)\big) + \phi_n(\epsilon, \lambda) + \psi_n(\epsilon, \delta, \lambda).$$

where $\phi_n(\epsilon, \lambda)$ and $\psi_n(\epsilon, \delta, \lambda)$ are negligible. More explictly

$$\phi_n(\epsilon, \lambda) = \mathbb{P}\Bigg(\exists\, k \in [\mathbf{0}, (n-1)\mathbf{1}], \quad \frac{\Gamma_j^{-1/\alpha} |f_k(U_j^{(n)})|}{\max_{m \in [\mathbf{0},(n-1)\mathbf{1}]} |f_m(U_j^{(n)})|} > \frac{\epsilon \lambda}{C_\alpha^{1/\alpha}} \text{ for at least 2 different } j\Bigg)$$

$$\le n^d \mathbb{P}\Bigg(\Gamma_j^{-1/\alpha} > \frac{b_n \epsilon \lambda}{C_\alpha^{1/\alpha} \|f\|_\alpha} \text{ for at least 2 different } j\Bigg).$$

For any $0 < \beta < \alpha$, we have

$$\begin{aligned}
\mathbb{E}\big[b_n^{-\beta} M_n^{\beta}\big] &= \int_0^\infty \mathbb{P}\big(b_n^{-1} M_n > \tau^{1/\beta}\big) d\tau \\
&\le \int_0^\infty \mathbb{P}\big(C_\alpha^{1/\alpha} \Gamma_1^{-1/\alpha} > \tau^{1/\beta}(1-\delta)\big) d\tau \\
&\quad + \int_0^\infty \phi_n(\epsilon, \tau^{1/\beta}) d\tau + \int_0^\infty \psi_n(\epsilon, \delta, \tau^{1/\beta}) d\tau \\
&:= T_1(\delta) + T_2^{(n)}(\epsilon) + T_3^{(n)}(\epsilon, \delta).
\end{aligned} \tag{13}$$

We prove that $T_2^{(n)}(\epsilon)$ and $T_3^{(n)}(\epsilon, \delta)$ go to 0 as $n \to \infty$, and

$$T_1(\delta) \to C$$

as $n \to \infty$ and $\delta \to 0+$.
Similarly, we prove the desired lower bound for $\mathbb{E}\big[b_n^{-\beta} M_n^{\beta}\big]$.

Thank you