Discrete rough paths and limit theorems

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Outline

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Some notation

Uniform partition of $[0,1]$: For $n \geq 1$ we set

$$t_k = \frac{k}{n}$$

Increment of a function: For $f : [0,1] \to \mathbb{R}^d$, we write

$$\delta f_{st} = f_t - f_s$$

Standard Gaussian measure: We set

$$\gamma = \mathcal{N}(0,1)$$

Example of CLT: Breuer-Major's theorem

Theorem 1.

Let

- $f \in L_2(\gamma)$ such that $\int f \, d\gamma = 0$
- B a 1-d fBm with Hurst parameter $\nu < \frac{1}{2}$

For $0 \le s \le t \le 1$ and $n \ge 1$, we set:

$$h^n_{st} = n^{-\frac{1}{2}} \sum_{s \le t_k < t} f\left(n^{\nu} \delta B_{t_k t_{k+1}}\right)$$

Then the following convergence holds true:

$$h^n \xrightarrow{f.d.d.} \sigma_f \, \delta W \quad \text{as} \quad n \to \infty$$

Note: This gives an example of CLT in highly dependent context

Discrete integrals (or weighted sums)

Motivation for the introduction of weights:

- Analysis of numerical schemes
- Parameter estimation based on quadratic variations
- Convergence of Riemann sums in rough contexts

Weighted sums (or discrete integrals):

For a function f and a process y, we set

$$\begin{aligned} \mathcal{J}_s^t(y; h^n) &= \sum_{s \le t_k < t} y_{t_k}\, h^n_{t_k t_{k+1}} \\ &= n^{-\frac{1}{2}} \sum_{s \le t_k < t} y_{t_k}\, f(n^{\nu} \delta B_{t_k t_{k+1}}) \end{aligned}$$

General question

Recall:

$$\mathcal{J}_s^t(y; h^n) = \sum_{s \le t_k < t} y_{t_k}\, h^n_{t_k t_{k+1}}$$

Question: Can we have

$$\text{CLT for } h^n \quad \Longrightarrow \quad \text{CLT for } \mathcal{J}_s^t(y; h^n) \ ?$$

Main message: Answer is YES if

- h^n comes from a rough path (e.g Breuer-Major context)
- y is a controlled process

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Rough path

Notation: We consider

- $\nu \in (0, 1/2]$, Hölder continuity exponent
- $\ell = \lfloor \frac{1}{\nu} \rfloor$, order of the rough path
- $\mathbb{R}^m$, state space for a process x

Rough path: Collection $\mathbf{x} = \{x^i; i \leq \ell\}$ such that

- $x^i = \{x^i_{st} \in (\mathbb{R}^m)^{\otimes i}; \, s, t \in [0, 1]\}$
- $x^i_{st} = \int_{s \leq s_1 < \cdots < s_i \leq t} dx_{s_1} \otimes \cdots \otimes dx_{s_i}$ (to be defined rigorously)
- We have

$$|x^i|_\nu \equiv \sup_{(u,v) \in \mathcal{S}_2} \frac{|x^i_{uv}|}{|v - u|^{\nu i}} < \infty$$

Controlled processes (incomplete definition)

Definition 2.

Let:

- $\ell = \lfloor \frac{1}{\nu} \rfloor$
- x a ν-rough path
- A family $\mathbf{y} = (y, y^{(1)}, \ldots, y^{(\ell-1)})$ of processes

We say that $\mathbf{y}$ is a process controlled by $\mathbf{x}$ if

$$\delta y_{st} = \sum_{i=1}^{\ell-1} y_s^{(i)} x_{st}^{i} + r_{st}, \quad \text{and} \quad |r_{st}| \lesssim |t-s|^{1-\nu}.$$

Remark: Typical examples of controlled process
$\hookrightarrow$ solutions of differential equations driven by x, or $g(x)$

Abstract transfer theorem: setting

Objects under consideration: Let

- α limiting regularity exponent. Typically $\alpha = \frac{1}{2}$ or $\alpha = 1$
- $\mathbf{x}$ rough path of order ℓ
- h^n such that uniformly in n:

$$|\mathcal{J}_s^t(x^i; h^n)|_{L_2} \leq K(t-s)^{\alpha+\nu i} \tag{1}$$

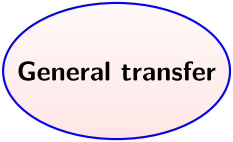
- $\mathbf{y}$ controlled process of order ℓ
- $(\omega^i,\, i \in \mathcal{I})$ family of processes independent of x
  $\hookrightarrow$ Typically ω_t^i = Brownian motion, or $\omega_t^i = t$

Abstract transfer theorem (1)

Recall: h^n satisfies:

$$|\mathcal{J}_s^t(x^i; h^n)|_{L_2} \leq K(t-s)^{\alpha+\nu i}$$

Illustration:



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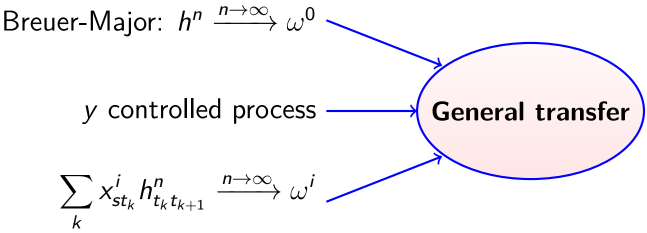

Abstract transfer theorem (1)

Recall: h^n satisfies:

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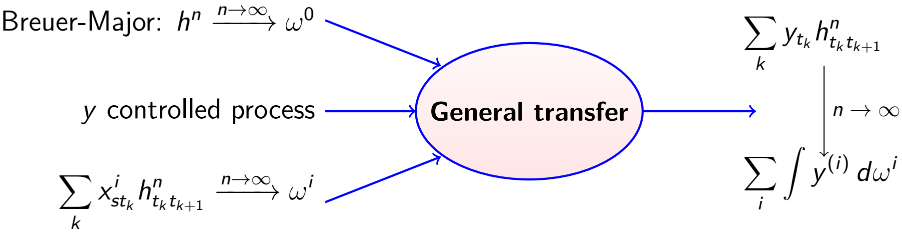

Abstract transfer theorem

Theorem 3.

We assume that (1) holds and:

1. For $i \leq \ell - 1$, as $n \to \infty$:

$$\mathcal{J}(x^i; h^n) \xrightarrow{f.d.d.} \omega^i.$$

2. One additional technical condition on $\int y \, d\omega^i$.

Then the following convergence holds true as $n \to \infty$:

$$\mathcal{J}(y; h^n) \xrightarrow{\text{f.d.d., stable}} \sum_{i=0}^{\ell-1} \int y^{(i)} \, d\omega^i.$$

Outline

Hermite rank

Definition 4.

Consider

- $\gamma = \mathcal{N}(0,1)$.
- $f \in L^2(\gamma)$ such that f is centered.

Then there exist:

- $d \geq 1$
- A sequence $\{c_q;\ q \geq d\}$

such that f admits an expansion on Hermite polynomials:

$$f = \sum_{q=d}^{\infty} c_q H_q.$$

The parameter d is called Hermite rank of f.

Breuer-Major with controlled weights (1)

Recall: For a fBm B and controlled process y we consider

$$\mathcal{J}_s^t(y; h^n) = n^{-\frac{1}{2}} \sum_{s \le t_k < t} y_{t_k} f(n^{\nu} \delta B_{t_k t_{k+1}})$$

Expected limit result: For W as in Breuer-Major,

$$\lim_{n\to\infty} \mathcal{J}_s^t(y; h^n) = \sigma_{d,f} \int_s^t y_u \, dW_u \qquad (2)$$

Unexpected phenomenon:
The limits of $\mathcal{J}_s^t(y; h^n)$ can be quite different from (2)

Breuer-Major with controlled weights (2)

Theorem 5.

For f smooth with Hermite rank d and y controlled we set

$$\mathcal{J}_s^t(y; h^{n,d}) = n^{-\frac{1}{2}} \sum_{s \le t_k < t} y_{t_k} f(n^{\nu} \delta B_{t_k t_{k+1}})$$

Then the following limits hold true:

1. If $d > \frac{1}{2\nu}$ then
$\mathcal{J}_s^t(y; h^{n,d}) \xrightarrow{(d)} c_{d,\nu} \int_s^t y_u \, dW_u$
2. If $d = \frac{1}{2\nu}$ then
$\mathcal{J}_s^t(y; h^{n,d}) \xrightarrow{(d)} c_{d,\nu} \int_s^t y_u \, dW_u + c_{2,d,\nu} \int_s^t y_u^{(d)} \, du$
3. If $1 \le d < \frac{1}{2\nu}$ then
$n^{-(\frac{1}{2} - \nu d)} \mathcal{J}_s^t(y; h^{n,d}) \xrightarrow{\mathbf{P}} c_d \int_s^t y_u^{(d)} \, du$

Breuer-Major with controlled weights (3)

Previous versions of Theorem 5:

- Obtained for $y = g(B)$ in a series of papers by Corcuera, Nualart, Nourdin, Podolskij, Réveillac, Swanson, Tudor

Improvements of our Theorem 5:

- One integrates w.r.t a general $f(n^{\nu} \delta B_{t_k t_{k+1}})$
  $\hookrightarrow$ with f smooth enough (vs. $f = H_d$)
- Results can be generalized to d-dim situations (vs. $d = 1$)
- General controlled weights y (vs. $y = g(B)$)

Other applications:

- Itô formulas in law, midpoint rules for rough paths
- Asymptotic behavior of p-variations, limits for num. schemes