Infinitely divisible random fields with long range dependence

Gennady Samorodnitsky
jointly with Zaoli Chen

Random sup measures

- $\mathcal{G}$ the collection of open sets in $\mathbb{R}^d$.

Random sup measures

- $\mathcal{G}$ the collection of open sets in $\mathbb{R}^d$.
- A sup measure: a map $m : \mathcal{G} \to [0, \infty]$ with $m(\emptyset) = 0$ and $m(\cup_\gamma G_\gamma) = \sup_\gamma m(G_\gamma)$ for any collection of open sets $\{G_\gamma\}$.

Random sup measures

- $\mathcal{G}$ the collection of open sets in $\mathbb{R}^d$.
- A sup measure: a map $m : \mathcal{G} \to [0, \infty]$ with $m(\emptyset) = 0$ and $m(\cup_\gamma G_\gamma) = \sup_\gamma m(G_\gamma)$ for any collection of open sets $\{G_\gamma\}$.
- The sup derivative $d^\vee m$ of a sup measure m:

$$d^\vee m(t) := \inf_{t \in G} m(G), \quad G \in \mathcal{G} .$$

- The sup integral of $f : \mathbb{R}^d \to [0, \infty]$:

$$i^{\vee} f(G) := \sup_{t \in G} f(t), \quad G \in \mathcal{G} .$$

- The sup integral of $f : \mathbb{R}^d \to [0, \infty]$:

$$i^\vee f(G) := \sup_{t \in G} f(t), \quad G \in \mathcal{G}\,.$$

- $m = i^\vee d^\vee m$.

- The sup integral of $f : \mathbb{R}^d \to [0, \infty]$:

$$i^\vee f(G) := \sup_{t \in G} f(t), \quad G \in \mathcal{G} .$$

- $m = i^\vee d^\vee m$.

- The space SM of sup measures with sup vague topology is compact and metrizable.

- The sup integral of $f : \mathbb{R}^d \to [0, \infty]$:

$$i^\vee f(G) := \sup_{t \in G} f(t), \quad G \in \mathcal{G} .$$

- $m = i^\vee d^\vee m$.

- The space SM of sup measures with sup vague topology is compact and metrizable.

- $\Longrightarrow$: a random sup measure.

Stable regenerative sets

- The β-stable regenerative set: the closure of the range of the β-subordinator, $0 < \beta < 1$.

Stable regenerative sets

- The β-stable regenerative set: the closure of the range of the β-subordinator, $0 < \beta < 1$.

- $R_{\beta_i}^{(i)}$: independent β_i-stable regenerative sets, $0 < \beta_i < 1$, $i = 1, \ldots, d$.

Stable regenerative sets

- The β-stable regenerative set: the closure of the range of the β-subordinator, $0 < \beta < 1$.
- $R^{(i)}_{\beta_i}$: independent β_i-stable regenerative sets, $0 < \beta_i < 1$, $i = 1, \ldots, d$.
- $v^{(i)} > 0$, $i = 1, \ldots, d$,

$$\tilde{R}_{\boldsymbol{\beta}} := \prod_{i=1}^{d} \left(v^{(i)} + R^{(i)}_{\beta_i} \right) .$$

- $\{\tilde{R}_{\boldsymbol{\beta},j}\}_{j\geq 1}$ independent.

- $\{\tilde{R}_{\boldsymbol{\beta},j}\}_{j\geq 1}$ independent.
- Suppose the shift vectors have different components.

- $\{\tilde{R}_{\boldsymbol{\beta},j}\}_{j\geq 1}$ independent.
- Suppose the shift vectors have different components.
- For any $m = 1, 2, \ldots$,

$$P(\cap_{j=1}^{m} \tilde{R}_{\boldsymbol{\beta},j} \neq \emptyset) = 0 \text{ or } 1\,,$$

the probability is 1 if and only if $m < \min_{i=1,\ldots,d}(1-\beta_i)^{-1}$.

The model

- A discrete time stationary random field $\mathbf{X} = \left(X_{\mathbf{t}},\, \mathbf{t} \in \mathbb{Z}^d\right)$.

The model

- A discrete time stationary random field $\mathbf{X} = \left(X_{\mathbf{t}},\, \mathbf{t} \in \mathbb{Z}^d\right)$.
- What does "memory" mean for random fields?

The model

- A discrete time stationary random field $\mathbf{X} = \left(X_{\mathbf{t}},\, \mathbf{t} \in \mathbb{Z}^d\right)$.
- What does "memory" mean for random fields?
- We consider a class of stationary symmetric α-stable (SαS) random fields, $0 < \alpha < 2$.

The model

- A discrete time stationary random field $\mathbf{X} = \big(X_{\mathbf{t}},\, \mathbf{t} \in \mathbb{Z}^d\big)$.
- What does "memory" mean for random fields?
- We consider a class of stationary symmetric α-stable (SαS) random fields, $0 < \alpha < 2$.
- There is a natural parametrization of "memory".

- d σ-finite, infinite measures on $\big(\mathbb{Z}^{\mathbb{Z}}, \mathcal{B}(\mathbb{Z}^{\mathbb{Z}})\big)$:

$$\mu_i = \sum_{k \in \mathbb{Z}} \pi_k^{(i)} P_k^{(i)} \, .$$

- d σ-finite, infinite measures on $\big(\mathbb{Z}^{\mathbb{Z}}, \mathcal{B}(\mathbb{Z}^{\mathbb{Z}})\big)$:

$$\mu_i = \sum_{k \in \mathbb{Z}} \pi_k^{(i)} P_k^{(i)} \,.$$

- For $i = 1, \ldots, d$, $P_k^{(i)}$: the law of an irreducible aperiodic null-recurrent Markov chain $(Y_n^{(i)})_{n \geq 0}$ on $\mathbb{Z}$, $Y_0^{(i)} = k \in \mathbb{Z}$.

- d σ-finite, infinite measures on $\left(\mathbb{Z}^{\mathbb{Z}}, \mathcal{B}(\mathbb{Z}^{\mathbb{Z}})\right)$:

$$\mu_i = \sum_{k\in\mathbb{Z}} \pi_k^{(i)} P_k^{(i)} .$$

- For $i = 1, \ldots, d$, $P_k^{(i)}$: the law of an irreducible aperiodic null-recurrent Markov chain $(Y_n^{(i)})_{n\geq 0}$ on $\mathbb{Z}$, $Y_0^{(i)} = k \in \mathbb{Z}$.

- $(\pi_k^{(i)})_{k\in\mathbb{Z}}$: the invariant measure satisfying $\pi_0^{(i)} = 1$.

- The first return time to the origin: for $\mathbf{x} = (\ldots, x_{-1}, x_0, x_1, x_2 \ldots) \in \mathbb{Z}^{\mathbb{Z}}$, $\varphi(\mathbf{x}) = \inf\{n \geq 1 : \; x_n = 0\}$.

- The first return time to the origin: for $\mathbf{x} = (\ldots, x_{-1}, x_0, x_1, x_2 \ldots) \in \mathbb{Z}^{\mathbb{Z}}$, $\varphi(\mathbf{x}) = \inf\{n \geq 1 : x_n = 0\}$.

- **The key assumption**:

$$P_0^{(i)}(\varphi > n) \in RV_{-\beta_i}, \text{ some } 0 < \beta_i < 1, \ i = 1, \ldots, d.$$

- The first return time to the origin: for $\mathbf{x} = (\dots, x_{-1}, x_0, x_1, x_2 \dots) \in \mathbb{Z}^{\mathbb{Z}}$, $\varphi(\mathbf{x}) = \inf\{n \geq 1 : \, x_n = 0\}$.
- **The key assumption**:

$$P_0^{(i)}(\varphi > n) \in RV_{-\beta_i}, \text{ some } 0 < \beta_i < 1, \, i = 1, \dots, d.$$

- Overall space:

$$(E, \mathcal{E}) = \left(\mathbb{Z}^{\mathbb{Z}} \times \cdot \times \mathbb{Z}^{\mathbb{Z}}, \, \mathcal{B}(\mathbb{Z}^{\mathbb{Z}}) \times \cdot \times \mathcal{B}(\mathbb{Z}^{\mathbb{Z}})\right) ,$$

$$\mu = \mu_1 \times \cdot \times \mu_d \, .$$

- Left shift operator on $\mathbb{Z}^{\mathbb{Z}}$:

$$T\big((\ldots, x_{-1}, x_0, x_1, x_2 \ldots)\big) = (\ldots, x_0, x_1, x_2, x_3 \ldots).$$

- Left shift operator on $\mathbb{Z}^{\mathbb{Z}}$:

$$T\big((\ldots, x_{-1}, x_0, x_1, x_2 \ldots)\big) = (\ldots, x_0, x_1, x_2, x_3 \ldots).$$

- A group action of $\mathbb{Z}^d$ on E: for $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$,

$$T^{\mathbf{n}}\mathbf{x} = (T^{n_1}\mathbf{x}^{(1)}, \ldots, T^{n_d}\mathbf{x}^{(d)}) \in E$$

  if $\mathbf{x} = (\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(d)})$

A stationary SαS random field:

$$X_{\mathbf{n}} = \int_E f \circ T^{\mathbf{n}}(\mathbf{x})\, M(d\mathbf{x}), \quad \mathbf{n} \in \mathbb{Z}^d\,,$$

A stationary SαS random field:

$$X_{\mathbf{n}} = \int_E f \circ T^{\mathbf{n}}(\mathbf{x}) \, M(d\mathbf{x}), \quad \mathbf{n} \in \mathbb{Z}^d \,,$$

- M: a SαS random measure on $(E, \mathcal{E})$ with control measure μ,

A stationary SαS random field:

$$X_{\mathbf{n}} = \int_E f \circ T^{\mathbf{n}}(\mathbf{x})\, M(d\mathbf{x}), \quad \mathbf{n} \in \mathbb{Z}^d\,,$$

- M: a SαS random measure on $(E, \mathcal{E})$ with control measure μ,

$$f(\mathbf{x}) = \mathbf{1}(\mathbf{x}^{(i)} \in A,\ i = 1, \ldots, d),\ \mathbf{x} = (\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(d)})\,.$$

A stationary SαS random field:

$$X_{\mathbf{n}} = \int_E f \circ T^{\mathbf{n}}(\mathbf{x})\, M(d\mathbf{x}), \quad \mathbf{n} \in \mathbb{Z}^d \,,$$

- M: a SαS random measure on $(E, \mathcal{E})$ with control measure μ,

$$f(\mathbf{x}) = \mathbf{1}(\mathbf{x}^{(i)} \in A,\ i = 1, \ldots, d),\ \mathbf{x} = (\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(d)})\,.$$

- $A = \{\mathbf{x} \in \mathbb{Z}^{\mathbb{Z}} :\ x_0 = 0\}$.

Extreme Value Theory for Random fields

Extremes of the random field over growing hypercubes

$$[\mathbf{0}, \mathbf{n}] = \left\{\mathbf{0} \leq \mathbf{k} \leq \mathbf{n}\right\}, \; \mathbf{n} \to \infty \,.$$

Extreme Value Theory for Random fields

Extremes of the random field over growing hypercubes

$$[\mathbf{0}, \mathbf{n}] = \{\mathbf{0} \le \mathbf{k} \le \mathbf{n}\},\ \mathbf{n} \to \infty\,.$$

- A random sup measure:

$$\eta_{\mathbf{n}}(B) := \max_{\mathbf{k}/\mathbf{n} \in B} X_{\mathbf{k}}, \quad B \in \mathcal{B}([0, \infty)^d)\,.$$

Extreme Value Theory for Random fields

Extremes of the random field over growing hypercubes

$$[\mathbf{0}, \mathbf{n}] = \big\{\mathbf{0} \le \mathbf{k} \le \mathbf{n}\big\},\ \mathbf{n} \to \infty\,.$$

- A random sup measure:

$$\eta_{\mathbf{n}}(B) := \max_{\mathbf{k}/\mathbf{n} \in B} X_{\mathbf{k}}, \quad B \in \mathcal{B}([0,\infty)^d)\,.$$

- Under what normalization $\eta_{\mathbf{n}}$ converges and what is the limit?

- Denote for $n = 1, 2, \dots$ and $i = 1, \dots, d$,

$$b_n^{(i)} = \left(\mu_i\left(\{\mathbf{x} : x_k = 0 \text{ for some } k = 0, 1, \dots, n\}\right)\right)^{1/\alpha}.$$

- Denote for $n = 1, 2, \dots$ and $i = 1, \dots, d$,

$$b_n^{(i)} = \big(\mu_i\left(\{\mathbf{x} : \, x_k = 0 \text{ for some } k = 0, 1, \dots, n\}\right)\big)^{1/\alpha}.$$

- $b_n^{(i)} \in \mathrm{Re}\big((1 - \beta_i)/\alpha\big)$

- Denote for $n = 1, 2, \dots$ and $i = 1, \dots, d$,

$$b_n^{(i)} = \big(\mu_i(\{\mathbf{x} : x_k = 0 \text{ for some } k = 0, 1, \dots, n\})\big)^{1/\alpha}.$$

- $b_n^{(i)} \in \mathrm{Re}\big((1 - \beta_i)/\alpha\big)$

- Set

$$b_{\mathbf{n}} = \prod_{i=1}^{d} b_{n_i}^{(i)}, \ \mathbf{n} = (n_1, \dots, n_d) \in \mathbb{N}_0^d.$$

Theorem: convergence in the space of sup measures

As $\mathbf{n} \to \infty$, weakly,

$$\frac{1}{b_{\mathbf{n}}}\eta_{\mathbf{n}} \Rightarrow \left(\frac{C_\alpha}{2}\right)^{1/\alpha} \eta_{\alpha,\boldsymbol{\beta}}\,.$$

Theorem: convergence in the space of sup measures

As $\mathbf{n} \to \infty$, weakly,

$$\frac{1}{b_{\mathbf{n}}}\eta_{\mathbf{n}} \Rightarrow \left(\frac{C_\alpha}{2}\right)^{1/\alpha} \eta_{\alpha,\boldsymbol{\beta}} \,.$$

$$C_\alpha = \left(\int_0^\infty x^{-\alpha} \sin x dx\right)^{-1} = \begin{cases} \frac{1-\alpha}{\Gamma(2-\alpha)\cos(\pi\alpha/2)} & \alpha \neq 1 \\ 2/\pi & \alpha = 1 \end{cases} .$$

The limit sup measure

$$\eta_{\alpha,\boldsymbol{\beta}}(B) = \sup_{t \in B} \sum_{j=1}^{\infty} U_{\alpha,j} \mathbf{1}_{\{\mathbf{t} \in \mathbf{V}_{\boldsymbol{\beta},j} + R_{\boldsymbol{\beta},j}\}}, \quad B \in \mathcal{B}([0,\infty)^d)\,.$$

The limit sup measure

$$\eta_{\alpha,\boldsymbol{\beta}}(B) = \sup_{t \in B} \sum_{j=1}^{\infty} U_{\alpha,j} \mathbf{1}_{\{\mathbf{t} \in \mathbf{V}_{\boldsymbol{\beta},j} + R_{\boldsymbol{\beta},j}\}}, \quad B \in \mathcal{B}([0,\infty)^d)\,.$$

- $(U_{\alpha,j}, \mathbf{V}_{\boldsymbol{\beta},j})_{j\geq 1}$: the points of a Poisson point process on $\mathbb{R} \times \mathbb{R}^d$, with mean measure

$$\alpha u^{-1-\alpha} du \prod_{i=1}^{d} (1-\beta_i) v_i^{-\beta_i} dv_i, \quad u, v_1, \ldots, v_d > 0\,.$$

- $\{R_{\boldsymbol{\beta}_i,j}\}_{j\geq 1}$: iid products of stable regenerative sets.

- $\{R_{\boldsymbol{\beta}_i,j}\}_{j\geq 1}$: iid products of stable regenerative sets.
- The limiting measure is stationary and self-similar.

- $\{R_{\boldsymbol{\beta}_i,j}\}_{j\geq 1}$: iid products of stable regenerative sets.
- The limiting measure is stationary and self-similar.
- It has the Fréchet distribution if and only if $\beta_i \leq 1/2$ for some $i = 1, \ldots, d$.