Semi-Long Range Dependence

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AMS Sectional Session on

Self-similarity and Long-range Dependence in Stochastic Processes

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Outline

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- Long-Range Dependence
- Semi-Long Range Dependence
- ARTFIMA time series
- Tempered linear processes
- Weak convergence to tempered fractional Brwonian motion
- Summary

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Long Range Dependence

- (General definition): A covariance stationary process $\{X_j\}$ with autocovariance function $\gamma(k)$ is asaid to have long memory if $\sum_{k\in\mathbb{Z}} |\gamma(k)| = \infty$.
- (Time domain): $\{X_j\}$ with autocovariance function $\gamma(k)$ is said to have long memory if $\gamma(k) \sim c_\gamma |k|^{2d-1}$ as $k \to \infty$, where $c_\gamma > 0$.
- (Frequency domain): $\{X_j\}$ with autocovariance function $\gamma(k)$ is said to have long memory if its spectral density $f(\nu)$ is bounded on $[\varepsilon, \pi]$ for any ε and $f(\nu) \sim c_f |\nu|^{-2d}$ as $\nu \to \infty$ and $-1/2 < d < 1/2$.
- (Great references for long range dependence) Stable non-Gaussian random processes (Samorodnitsky and Taqqu [10]), Large sample inference for long memory processes (Giraitis, Koul, Surgagilis [5]), Stochastic processes and long range dependence (Samorodnitsky [9]).

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- Giraitis et al. [1] termed such behavior as "semi long memory" (Analogously to the term semi heavy-tails applied to probability densities asymptotically behaving as $x^{-\alpha}\beta^{-x}(\alpha > 0, \beta > 0)$ when $x \to \infty$).
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Why semi-long range dependence?

- Kolmogorov [4] proposed a model for the energy spectrum of turbulence in the inertial range, predicting that the spectrum $f(k)$ would follow a power law $f(k) \propto k^{-5/3}$ where

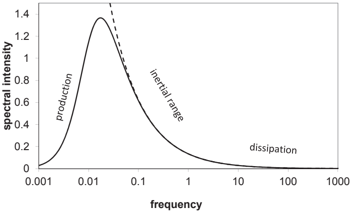

Figure: Kolmogorow spectral density (solid line) and power law approximation in the inertial range (dotted line). Large eddies are produced in the low frequency range.

- Von Kármán [11] in 1948 published one of the earliest research paper to discuss about appropriate spectral density that covers both production and inertial ranges.

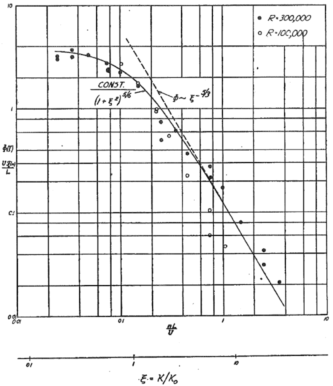

FIGURE 2

Comparison of observed and computed values of the frequency spectrum.

Figure: The Figure shows that the appropriate spectral density would be proportional to $\left(1+\omega^2\right)^{-5/6}$ based on the empirical works of Von Kármán.

Tempered fractional difference operator

- The tempered fractional difference operator is defined by:

$$\Delta^{d,\lambda} f(t) = (I - e^{-\lambda} B)^d f(t) = \sum_{j=0}^{\infty} \omega_j^{d,\lambda} f(t-j)$$

where $d > 0$, $d \notin \mathbb{Z}$, $\lambda > 0$, $Bf(t) = f(t-1)$ is the shift operator, and

$$\omega_j^{d,\lambda} := (-1)^j \binom{d}{j} e^{-\lambda j} \quad \text{where} \quad \binom{d}{j} = \frac{\Gamma(1+d)}{j!\Gamma(1+d-j)}$$

ARTFIMA time series

Definition

The discrete time stochastic process $\{X_t\}_{t\in\mathbb{Z}}$ follows an autoregressive tempered fractional integrated moving average time series, denoted by ARTFIMA(p, d, λ, q), if

$$Y_t = \Delta^{d,\lambda} X_t = (I - e^{-\lambda} B)^d X_t \tag{0.1}$$

follows the ARMA(p, q) model

$$Y_t - \sum_{j=1}^{p} \phi_j Y_{t-j} = Z_t + \sum_{i=1}^{q} \theta_i Z_{t-i} \tag{0.2}$$

where $\{Z_t\}_{t\in\mathbb{Z}}$ is a white noise sequence (i.i.d. with $\mathbb{E}[Z_t] = 0$ and $\mathbb{E}[Z_t^2] = \sigma^2$), $d \notin \mathbb{Z}$, $\lambda > 0$, and $\Phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \ldots - \phi_p z^p$, and $\Theta(z) = 1 + \theta_1 z + \theta_2 z^2 + \ldots + \theta_q z^q$ are polynomials of degrees $p, q \geq 0$ with no common zeros.

- The spectral density of $X_{p,d,\lambda,q}$ is given by

$$h(x) = \frac{1}{2\pi}\left|\frac{\Theta(e^{-ix})}{\Phi(e^{-ix})}\right|^2 \left(1-2e^{-\lambda}\cos x + e^{-2\lambda}\right)^{-d}, \qquad -\pi \le x \le \pi.$$

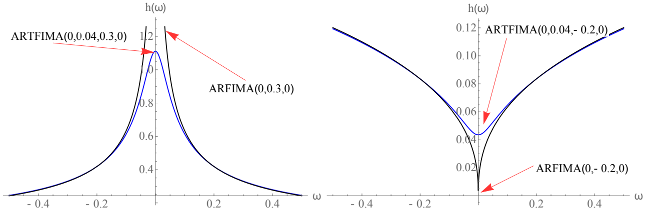

Figure: The left panel shows the spectral density of ARTFIMA (blue curve), does not blow up to infinity in contrast to the untempered case. The right panel shows the value of the spectral density of ARTFIMA at the origin is a non-zero finite number while for the ARFIMA case, the spectral density is zero at the origin.

- The covariance function of $X_{0,d,\lambda,0}$ is given by

$$\gamma_{d,\lambda}(k) = \mathbb{E}X_{0,d,\lambda,0}(0)X_{0,d,\lambda,0}(k) = \frac{e^{-\lambda k}\Gamma(d+k)}{\Gamma(d)\Gamma(k+1)}\, {}_2F_1(d, k+d; k+1;\ e^{-2\lambda}),$$

where ${}_2F_1(a, b; c; z)$ is the Gauss hypergeometric function. Moreover,

$$\sum_{k\in\mathbb{Z}} |\gamma_{d,\lambda}(k)| < \infty, \qquad \sum_{k\in\mathbb{Z}} \gamma_{d,\lambda}(k) = (1-\ e^{-\lambda})^{-2d} \tag{0.3}$$

and

$$\gamma_{d,\lambda}(k) \sim Ak^{d-1}\ e^{-\lambda k}, \quad k \to \infty, \quad \text{where} \quad A = (1-\ e^{-2\lambda})^{-d}\Gamma(d)^{-1}.$$

- Let $\mathbf{X} = (X_1, \ldots, X_N)$ be a realization of the ARTFIMA(p, d, λ, q) time series with sample size N and consider the periodogram

$$I_{\mathbf{X}}(\nu) := \frac{1}{2\pi N} \left| \sum_{t=1}^{N} X_t e^{it\nu} \right|^2. \tag{0.4}$$

Define

$$Q_{\mathbf{X}}(\theta) := \int_{-\pi}^{\pi} \frac{I_{\mathbf{X}}(\nu)}{K(\nu, \theta)} d\nu \tag{0.5}$$

and

$$D_N(\mathbf{X}, \sigma, \theta) := \frac{1}{2\sigma^2} Q_{\mathbf{X}}(\theta) + \log \sigma. \tag{0.6}$$

Let σ_0 and θ_0 denote the true parameter values of $\sigma \in (0, \infty)$ and $\theta = (\phi_1, \ldots, \phi_p, \theta_1, \ldots, \theta_q, d, \lambda) \in \Xi$, respectively, where $\Xi = \mathbb{R}^{p+q+1} \times (0, \infty)$. Define $\Omega = (0, \infty) \times \Xi$.

- The *Whittle estimators* of σ_0 and θ_0 based on $\mathbf{X} = (X_1, \ldots, X_N)$ are defined by

$$(\bar{\sigma}_N, \bar{\theta}_N) := \arg\min\{D_N(\mathbf{X}, \sigma, \theta) : (\sigma, \theta) \in \Omega\}, \qquad (0.7)$$

so that $\bar{\theta}_N = \arg\min\{Q_{\mathbf{X}}(\theta) : \theta \in \Xi\}$ and $\bar{\sigma}_N^2 = Q_{\mathbf{X}}(\bar{\theta}_N)$.

Theorem

The Whittle estimators (0.7) *over the compact parameter space* Ω_0 *are strongly consistent. That is,*

$$\lim_{N \longrightarrow \infty} \bar{\theta}_N = \theta_0 \qquad a.s.$$

and

$$\lim_{N \longrightarrow \infty} \bar{\sigma}_N^2(\bar{\theta}_N) = \sigma_0^2 \qquad a.s.$$

Theorem

The Whittle estimators (0.7) *over the compact parameter space* Ω_0 *are asymptotically normal. That is,* $N^{1/2}(\bar{\theta}_N - \theta_0)$ *converges in distribution to a Gaussian random vector with zero mean vector and covariance matrix* $\mathbf{W}^{-1}$ *where*

$$\mathbf{W} = \frac{1}{4\pi} \int_{-\pi}^{\pi} \left\{ \frac{\partial \log \kappa(\nu, \theta_0)}{\partial \theta} \right\} \left\{ \frac{\partial \log \kappa(\nu, \theta_0)}{\partial \theta} \right\}' d\nu. \tag{0.8}$$

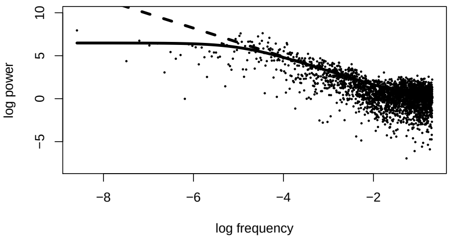

Figure: Figure 4 shows the periodogram, fitted ARTFIMA$(0, d, \lambda, 0)$ and ARFIMA$(0, d, 0)$ spectral density function for a data set from Saginaw Bay. We set $d = 5/6$ (according to Kolmogorov scaling) and this resulted in the parameter fit $\lambda = 0.045$ using the Whittle estimator. Both models fit the data for moderate frequencies, but the ARFIMA model shows a lack of fit for low frequencies. Hence we consider the ARFIMA model to be mis-specified, since its pure power law spectrum does not fit the data at low frequencies, and more importantly, the ARFIMA model with $d = 5/6$ is not stationary.

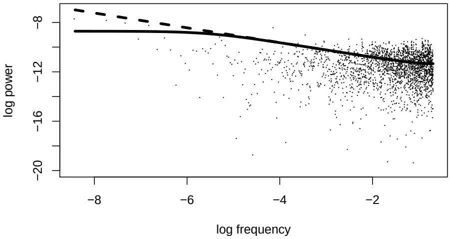

Figure: The adjusted closing price C_t for AMZN stock from 1/3/2000 to 12/19/2017 ($n = 4520$) was used to compute the squared log returns $X_t = \left(\ln(C_t/C_{t-1})\right)^2$. An ARTFIMA model was fitted with parameters $d = 0.3$ and $\lambda = 0.025$. Figure 5 shows that the resulting model spectral density provides a reasonable fit to the periodogram, which follows a power law at moderate frequencies, but levels off at low frequencies. The untempered ARFIMA model fails to fit the periodogram at low frequencies, hence we consider the ARTFIMA model fit to be superior, evidence of semi-long range dependence in the squared log-returns.

Tempered linear process

- The ARTFIMA time series can be extended to a general class of discrete stochastic processes which is called tempered linear processes. A tempered linear processes is a stochastic processes with moving averages

$$X_{d,\lambda}(t) = \sum_{k=0}^{\infty} e^{-\lambda k} b_d(k) \zeta(t-k), \qquad t \in \mathbb{Z} \tag{0.9}$$

in i.i.d. innovation process $\{\zeta(t)\}$ with coefficients $b_d(k)$ regularly varying at infinity as k^{d-1}, viz.

$$b_d(k) \sim \frac{c_d}{\Gamma(d)} k^{d-1}, \qquad k \to \infty, \qquad c_d \neq 0, \quad d \neq 0 \tag{0.10}$$

where $d \in \mathbb{R}$ is a real number, $d \neq -1, -2, \ldots$ and $\lambda > 0$ is tempering parameter.

Functional limit theorem

Question: *Assume that the tempering parameter* $\lambda \equiv \lambda_N$ *may depend on N so that it remains bounded as N increases and following limit exists:*

$$\lim_{N\to\infty} N\lambda_N = \lambda_* \in [0,\infty].$$

What is the weak convergence limit of scaled partial sums of tempered linear process $X_{d,\lambda}$ *in* (0.9) *?*

- Strongly tempered process Let $\lambda_* = \infty$ and $d \in \mathbb{R} \setminus \mathbb{N}_-$. Then

$$N^{-\frac{1}{2}} \lambda_N^d S_N^{d,\lambda_N}(t) \xrightarrow{\text{fdd}} \sigma B(t),$$

where B is a standard Brownian motion and $\sigma > 0$ some constant.

- If $\mathbb{E}|\zeta(0)|^p < \infty$ for some $p > 2$ then

$$N^{-\frac{1}{2}} \lambda_N^d S_N^{d,\lambda_N}(t) \xrightarrow{D[0,1]} \sigma B(t). \tag{0.11}$$

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- Weakly tempered process Let $\lambda_* = 0$ and $H = d + \frac{1}{2} \in (0,1)$. Then

$$N^{-H} S_N^{d,\lambda_N}(t) \xrightarrow{\text{fdd}} \Gamma(d+1)^{-1} B_{H,0}(t),$$

where $B_{H,0} = B_H$ is a multiple of fractional Brownian motion.

- If either $1/2 < H < 1$, or $0 < H < 1/2$ and $\mathbb{E}|\zeta(0)|^p < \infty\ (\exists p > 1/H)$ hold, then $\xrightarrow{\text{fdd}}$ can be replaced by $\xrightarrow{D[0,1]}$.

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- Moderately tempered process Let $\lambda_* \in (0, \infty)$ and $H = d + \frac{1}{2} > 0$. Then

$$N^{-H} S_N^{d,\lambda_N}(t) \xrightarrow{\text{fdd}} \Gamma(d+1)^{-1} B^{II}_{H,\lambda_*}(t),$$

where B^{II}_{H,λ_*} is a stochastic processes which is called tempered fractional Brownian motion of the second kind (TFBM II).

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- A tempered fractional Brownian motion of the second kind (TFBMII) is a Gaussian process with the stochastic integral

$$B^{II}_{H,\lambda}(t) \quad := \quad \int_{\mathbb{R}} h_{H,\lambda}(t;y)B(dy),$$

where

$$h_{H,\lambda}(t;y) = (t-y)_+^{H-\frac{1}{2}} e^{-\lambda(t-y)_+} - (-y)_+^{H-\frac{1}{2}} e^{-\lambda(-y)_+} + \lambda \int_0^t (s-y)_+^{H-\frac{1}{2}} e^{-\lambda(s-y)_+}\, ds.$$

- TFBM II $B^{II}_{H,\lambda}$ is well-defined for any $t \geq 0$ and $H > 0, \lambda > 0$.
- TFBM II $B^{II}_{H,\lambda}$ has stationary increments. Moreover, it satisfies the following scaling property:

$$\left\{B^{II}_{H,\lambda}(ct)\right\}_{t\in\mathbb{R}} \overset{f.d.d.}{=} \left\{c^H B^{II}_{H,c\lambda}(t)\right\}_{t\in\mathbb{R}}, \qquad \forall\, c > 0.$$

- TFBM II $B^{II}_{H,\lambda}$ in (0.12) has a.s. continuous paths if $H > 0$.

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Thank You!

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for $0 < \alpha \leq 2, 0 < H < 1, \lambda > 0$ and symmetric α-stable Lévy process M_α.

- For $\lambda = 0$ both TFSM and TFSM II agree with fractional stable motion.

- For any $\lambda > 0$ and $\kappa > 0$, we define the (positive and negative) tempered fractional integrals (TFI) of a function f by

$$\mathbb{I}_{\pm}^{\kappa,\lambda} f(y) = \frac{1}{\Gamma(\kappa)} \int_{\mathbb{R}} f(u)(u-y)_{\pm}^{\kappa-1} e^{-\lambda(u-y)_{\pm}} du.$$

- For $0 < \kappa < 1$ and $\lambda > 0$, we define the (positive and negative) tempered fractional derivatives (TFD) of a function f by

$$\mathbb{D}_{\pm}^{\kappa,\lambda} f(y) = \lambda^{\kappa} f(y) + \frac{\kappa}{\Gamma(1-\kappa)} \int_{\mathbb{R}} \frac{f(y)-f(u)}{(y-u)_{\pm}^{\kappa+1}} e^{-\lambda(y-u)_{\pm}} du.$$

- For any $\lambda > 0$ and $\kappa > 0$, we define the (positive and negative) tempered fractional integrals (TFI) of a function f by

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- The variance and covariance of TFBM II $B^{II}_{H,\lambda}$ $\left(H > 0, \lambda > 0\right)$ has the form

$$C_t^2 = \mathbb{E}\left[\left(B^{II}_{H,\lambda}(t)\right)^2\right] = \frac{1}{2\pi}\int_{\mathbb{R}}\left|\frac{e^{\mathrm{i}\omega t}-1}{\mathrm{i}\omega}(\lambda+\mathrm{i}\omega)^{\frac{1}{2}-H}\right|^2 d\omega$$
$$= \frac{\Gamma(-1/2)\Gamma(H)}{\pi\Gamma(H-\frac{1}{2})\lambda^{2H}}\left[1 - {}_2F_3\left(\{1,-1/2\},\{1-H,1/2,1\},\frac{\lambda^2 t^2}{4}\right)\right]$$
$$+ 2^{1-2H}\pi^{-1}\Gamma(H)t^{2H}{}_2F_3\left(\{1,H-\frac{1}{2}\},\{1,H+1,H+\frac{1}{2}\},\frac{\lambda^2 t^2}{4}\right), \tag{0.12}$$

and

$$\mathrm{Cov}\left[B^{II}_{H,\lambda}(t), B^{II}_{H,\lambda}(s)\right] = \frac{1}{2}\left[C_t^2 + C_s^2 - C_{t-s}^2\right], \qquad s,t\in\mathbb{R}, \tag{0.13}$$

where C_t^2 is given in (0.12) and ${}_2F_3$ is the generalized hypergeometric function.

- We can extend the definition of tempered fractional derivatives to a suitable class of functions in $L^2(\mathbb{R})$. For any $\kappa > 0$ and $\lambda > 0$ we may define the fractional Sobolev space

$$W^{\kappa,2}(\mathbb{R}) := \{f \in L^2(\mathbb{R}) : \int_{\mathbb{R}} (\lambda^2 + \omega^2)^{\kappa} |\hat{f}(\omega)|^2 \, d\omega < \infty\},$$

which is a Banach space with norm $\|f\|_{\kappa,\lambda} = \|(\lambda^2 + \omega^2)^{\kappa/2} \hat{f}(\omega)\|_2$.

- Let $\kappa > 0$ and $f \in L_1(\mathbb{R})$ (or $L_2(\mathbb{R})$). Then $\mathbb{I}_{-}^{\kappa,\lambda} f(t)$ has the Fourier transform

$$\widehat{\mathbb{I}_{\pm}^{\kappa,\lambda} f}(\omega) = \hat{f}(\omega)(\lambda \pm i\omega)^{-\kappa}$$

for $\lambda > 0$. For $f \in W^{\kappa,2}(\mathbb{R})$,

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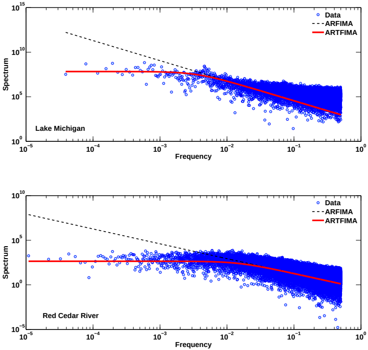

Figure: Two additional examples, along with fitted ARTFIMA spectrum (thick line), and Kolmogorov spectrum with slope $-5/3$ (thin dashed line). The data were collected in Lake Michigan (left panel) and the Red Cedar River (right panel).