Time fractional Cauchy problems in compact manifolds

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October 20-21, 2018. University of Michigan, Ann Arbor

- Lévy processes - Subordinators - Hitting times

- Fractional Cauchy problem on a compact manifold $\mathcal{M}$

- The case $\mathcal{M} = \mathbb{S}^2_1$, the unit sphere

- Random fields on $\mathbb{S}^2_1$

- Time dependent coordinates changed random fields on $\mathbb{S}^2_1$

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fractional Cauchy problem on $D \subseteq \mathbb{R}^d$

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Lévy processes

If $X(t)$, $t > 0$ is a Lévy processes, then the characteristic function

$$\mathbb{E}e^{i\boldsymbol{\xi}\cdot X(t)} = \int e^{i\boldsymbol{\xi}\cdot\mathbf{x}} p(t;\mathbf{x})d\mathbf{x} = \widehat{p}(t;\boldsymbol{\xi}) = e^{-t\Psi(\xi)} \tag{1}$$

is written in terms of the following Fourier symbol (Lévy - Khintchine)

$$\Psi(\boldsymbol{\xi}) = i\mathbf{b}\cdot\boldsymbol{\xi} + \boldsymbol{\xi}\cdot M\boldsymbol{\xi} - \int_{\mathbb{R}^d - \{0\}} \left(e^{i\boldsymbol{\xi}\cdot\mathbf{y}} - 1 - i\boldsymbol{\xi}\cdot\mathbf{y}\mathbf{1}_{(|\mathbf{y}|\le 1)}\right)\mu(d\mathbf{y}) \tag{2}$$

where $\mathbf{b} \in \mathbb{R}^d$, M is a non-negative definite symmetric $d \times d$ matrix and μ is a Lévy measure on $\mathbb{R}^d - \{0\}$, that is a Borel measure on $\mathbb{R}^d - \{0\}$ such that

$$\int (|\mathbf{y}|^2 \wedge 1)\mu(d\mathbf{y}) < \infty \quad \text{or equivalently} \quad \int \frac{|\mathbf{y}|^2}{1+|\mathbf{y}|^2}\mu(d\mathbf{y}) < \infty. \tag{3}$$

Subordinators

If D_t, $t > 0$ is a subordinator, then its Lévy symbol is written as

$$\eta(\xi) = ib\xi + \int_0^\infty \left(e^{i\xi y} - 1\right) \mu(dy) \tag{4}$$

where $b \geq 0$ and the Lévy measure μ satisfies the following requirements: $\mu(-\infty, 0) = 0$ and

$$\int (y \wedge 1)\mu(dy) < \infty \quad \text{or equivalently} \quad \int \frac{y}{1+y}\mu(dy) < \infty. \tag{5}$$

The Laplace exponent

$$\psi(\xi) = -\eta(i\xi) = b\xi + \int_0^\infty \left(1 - e^{-\xi y}\right) \mu(dy) \tag{6}$$

is a Bernstein function (that is $f : (-1)^k f^{(k)}(x) \leq 0$ for all $x \geq 0$).

Semigroups

Let us consider the Lévy process $X(t)$, $t > 0$, with associated Feller semigroup

$$P_t\, f(\mathbf{x}) = \mathbb{E} f(X(t) - \mathbf{x}) = \int_{\mathbb{R}^d} f(y-x) p(t;y) dy = \int_{\mathbb{R}^d} f(y) p(t; y+x) dy$$

solving $\partial_t u = \mathcal{A} u$, $u_0 = f$. In particular, P_t is a positive contraction semigroup (i.e. $0 \leq f \leq 1 \Rightarrow 0 \leq P_t f \leq 1$ and $P_{t+s} = P_t P_s$) on $C_\infty(\mathbb{R}^d)$ such that (Feller semigroup)

- $P_t(C_\infty(\mathbb{R}^d)) \subset C_\infty(\mathbb{R}^d)$, $t > 0$ (P_t is invariant),
- $P_t f \to f$ as $t \to 0$ for all $f \in C_\infty(\mathbb{R}^d)$ under the sup-norm (P_t is a strongly continuous contraction semigroup on the Banach space $(C_\infty(\mathbb{R}^d), \|\cdot\|_\infty)$).

Pseudo-differential operators

We are able to compute the semigroup and its generator as pseudo-differential operators. We say that $\mathcal{A}$ is the infinitesimal generator of $X(t)$, $t > 0$ and the following representation holds

$$\mathcal{A}f(\mathbf{x}) = -\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-i\boldsymbol{\xi}\cdot\mathbf{x}}\Psi(\boldsymbol{\xi})\widehat{f}(\boldsymbol{\xi})\boldsymbol{d\xi} \tag{7}$$

(where $\widehat{f}(\boldsymbol{\xi}) = \int_{\mathbb{R}^d} e^{i\boldsymbol{\xi}\cdot\mathbf{x}}f(\mathbf{x})\mathbf{dx}$) for all functions in the domain

$$D(\mathcal{A}) = \left\{ f \in L^2(\mathbb{R}^d, \mathbf{dx}) \, : \, \int_{\mathbb{R}^d} \Psi(\boldsymbol{\xi})|\widehat{f}(\boldsymbol{\xi})|^2 \boldsymbol{d\xi} < \infty \right\} \tag{8}$$

We say that P_t is a pseudo-differential operator with symbol $\exp(-t\Psi)$ and, $-\Psi$ is the Fourier multiplier (or Fourier symbol) of $\mathcal{A}$,

$$\widehat{\mathcal{A}f}(\boldsymbol{\xi}) = -\Psi(\boldsymbol{\xi})\widehat{f}(\boldsymbol{\xi}) \quad \Leftrightarrow \quad \widehat{P_t f}(\boldsymbol{\xi}) = e^{-t\Psi(\boldsymbol{\xi})}\widehat{f}(\boldsymbol{\xi}). \tag{9}$$

Bochner subordination rule

Let $B(t) = (B_1(t), \ldots, B_d(t))$, $t > 0$ be a BM on $\mathbb{R}^d$ with $\Psi(\xi) = |\boldsymbol{\xi}|^2$

Let D_t, $t > 0$ be a stable subordinator on $[0, +\infty)$ with symbol $\psi(\xi) = \xi^\alpha$

What about the symbol and the generator of the process

$$Z(t) = B(D_t), \quad t > 0 \tag{10}$$

We get

$$\psi \circ \Psi(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^{2\alpha}, \quad \alpha \in (0, 1] \tag{11}$$

and

$$\mathcal{A} = -\left(-\Delta\right)^{\alpha} \tag{12}$$

is the fractional Laplacian.

...fractional Laplacian

the fractional power of the Laplace operator can be expressed as

$$-(-\Delta)^{\alpha} f(\mathbf{x}) = C_d(\alpha)\,\text{p.v.}\int_{\mathbb{R}^d} \frac{f(\mathbf{y}) - f(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|^{2\alpha+d}}\mathbf{dy} = C_d(\alpha)\,\text{p.v.}\int_{\mathbb{R}^d} \frac{f(\mathbf{x}+\mathbf{y}) - f(\mathbf{x})}{|\mathbf{y}|^{2\alpha+d}}\mathbf{dy} \tag{13}$$

where "p.v." stands for the "principal value" of the singular integrals above near the origin. For $\alpha \in (0,1)$, the fractional Laplace operator can be defined, for $f \in \mathscr{S}$ (the space of rapidly decaying C_∞ functions), as follows

$$\begin{aligned}-(-\Delta)^{\alpha} f(\mathbf{x}) =& \frac{C_d(\alpha)}{2}\int_{\mathbb{R}^d} \frac{f(\mathbf{x}+\mathbf{y}) + f(\mathbf{x}-\mathbf{y}) - 2f(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|^{2\alpha+d}}\mathbf{dy} \\ =& \frac{C_d(\alpha)}{2}\int_{\mathbb{R}^d} \frac{f(\mathbf{x}+\mathbf{y}) + f(\mathbf{x}-\mathbf{y}) - 2f(\mathbf{x})}{|\mathbf{y}|^{2\alpha+d}}\mathbf{dy}, \quad \forall\, \mathbf{x} \in \mathbb{R}^d. \end{aligned} \tag{14}$$

The beautiful formula

From the Bernstein function

$$x^{\alpha} = \int_0^{\infty} \left(1 - e^{-sx}\right) \frac{\alpha}{\Gamma(1-\alpha)} \frac{ds}{s^{\alpha+1}} \tag{15}$$

and the semigroup $P_s = e^{s\Delta}$, we define

$$-(-\Delta)^{\alpha} f(x) = \int_0^{\infty} \left(P_s f(x) - f(x)\right) \frac{\alpha}{\Gamma(1-\alpha)} \frac{ds}{s^{\alpha+1}}. \tag{16}$$

The beautiful formula

From the Bernstein function

$$x^\alpha = \int_0^\infty \left(1 - e^{-sx}\right) \frac{\alpha}{\Gamma(1-\alpha)} \frac{ds}{s^{\alpha+1}} \tag{17}$$

and the semigroup $P_s = e^{s\Delta}$, we define

$$-(-\Delta)^\alpha f(x) = \int_0^\infty (P_s f(x) - f(x)) \frac{\alpha}{\Gamma(1-\alpha)} \frac{ds}{s^{\alpha+1}}. \tag{18}$$

Let ψ be the Laplace exponent of the subordinator D_t, $t > 0$, then

$$-\psi(-\Delta) f(x) = \int_0^\infty (P_s f(x) - f(x))\, \mu(ds) \tag{19}$$

is the infinitesimal generator of the subordinated process $Z(t) = B(D_t)$, $t > 0$

Fourier multiplier

From

$$-\psi(-\mathcal{A})f(x) = \int_0^\infty \left(P_s f(x) - f(x)\right)\mu(ds) \tag{20}$$

where $P_s = \exp s\mathcal{A}$, we have that

$$\begin{aligned}
-\psi\widehat{(-\mathcal{A})f}(\xi) &= \int_0^\infty \left(\widehat{P_s f}(\xi) - \widehat{f}(\xi)\right)\mu(ds) \\
&= \int_0^\infty \left(e^{-s\Psi(\xi)}\widehat{f}(\xi) - \widehat{f}(\xi)\right)\mu(ds) \\
&= -\int_0^\infty \left(1 - e^{-s\Psi(\xi)}\right)\mu(ds)\widehat{f}(\xi) = -\psi \circ \Psi(\xi)\widehat{f}(\xi)
\end{aligned}$$

where

$$\psi(\xi) = \int_0^\infty \left(1 - e^{-s\xi}\right)\mu(ds) \tag{21}$$

Inverse to a stable subordinator

Let D_t^α be a stable subordinator on $[0,+\infty)$ with symbol $\psi(\xi)=\xi^\alpha$, $\alpha\in(0,1)$. Then, E_t^α such that

$$E_t^\alpha = \inf\{s\geq 0 \,:\, D_s^\alpha > t\} \tag{22}$$

or such that

$$Pr\{E_t^\alpha < x\} = Pr\{D_x^\alpha > t\} \tag{23}$$

is the hitting time of or the inverse to D_t. Its probability law solves

$$\partial_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{du(x,r)}{dr}\frac{dr}{(t-r)^\alpha} = -\partial_x u(x,t) \tag{24}$$

The Caputo derivative ∂_t^α has Laplace transform

$$\int_0^\infty e^{-st}\partial_t^\alpha g(t)\, ds = s^\alpha \tilde{g}(s) - s^{\alpha-1}g(0) \tag{25}$$

Time-changed Brownian motion on $\mathbb{R}$

$$X = B(E_t^{\alpha}), \quad t > 0 \tag{26}$$

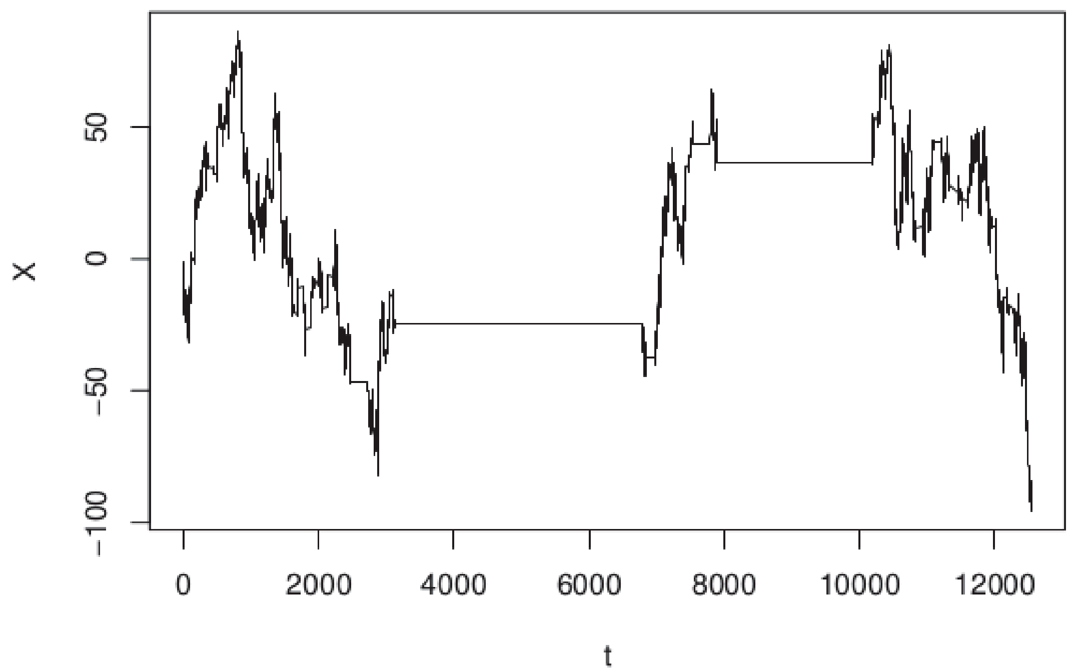


Figure 1: ”from the web page of Prof. Mark M. Meerschaert ”

Compact manifold and Laplacian

Let $\mathcal{M}$ be a smooth connected Riemannian manifold of dimension $n \geq 1$ with Riemannian metric g. The associated Laplace-Beltrami operator $\Delta = \Delta_{\mathcal{M}}$ in $\mathcal{M}$ is an elliptic, second order, differential operator defined in the space $C_0^\infty(\mathcal{M})$. In local coordinates, this operator is written as

$$\Delta = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(g^{ij} \sqrt{g} \frac{\partial}{\partial x_j} \right) \tag{27}$$

where $\{g_{ij}\}$ is the matrix of the Riemannian metric, $\{g^{ij}\}$ and g are respectively the inverse and the determinant of $\{g_{ij}\}$.

The heat kernel is the transition density of a diffusion process on $\mathcal{M}$ which is a Brownian motion generated by Δ.

Cauchy problem on $\mathcal{M}$

For any $y \in \mathcal{M}$, the heat kernel $p(x,y,t)$ is the fundamental solution to the heat equation

$$\partial_t u = \Delta u \tag{28}$$

with initial point source at y. Furthermore, $p(x,y,t)$ defines an integral kernel of the heat semigroup $P_t = e^{-t\Delta}$ and $p(x,y,t)$ is the transition density of a diffusion process on $\mathcal{M}$ which is a Brownian motion generated by Δ. If $\mathcal{M}$ is compact, then P_t is a compact operator on $L^2(\mathcal{M})$. By the general theory of compact operators, the transition density (heat kernel) $p(x,y,t)$ can be represented as a series expansion in terms of the eigenfunctions of $-\Delta$, say $\phi_j \in C^\infty$ and

$$0 \le \lambda_1 < \lambda_2 \le \ldots \uparrow +\infty$$

such that

$$p(x,y,t) = \sum_{j=1}^{\infty} e^{-\lambda_j t}\phi_j(x)\phi_j(y) \quad \forall\, t > 0 \tag{29}$$

Fractional Cauchy problem on $\mathcal{M}$ - 1/2

Let

$$H^s(\mathcal{M}) = \left\{ f \in L^2(\mathcal{M}) : \sum_{l=0}^{\infty} (\lambda_l)^{2s} \left(\int_{\mathcal{M}} \phi_l(y)\, f(y) \mu(dy) \right)^2 < \infty \right\}. \quad (30)$$

We say that Δf exists in the strong sense if it exists pointwise and is continuous in $\mathcal{M}$.

Similarly, we say that $\partial_t^{\beta} f(t, \cdot)$ exists in the strong sense if it exists pointwise and is continuous for $t \in [0, \infty)$. One sufficient condition is that f is a C^1 function on $[0, \infty)$ with $|f'(t, \cdot)| \leq c\, t^{\gamma - 1}$ for some $\gamma > 0$. Then, the Caputo fractional derivative $\partial_t^{\beta} f(t, \cdot)$ of f exists for every $t > 0$ and the derivative is continuous in $t > 0$.

Fractional Cauchy problem on $\mathcal{M}$ - 2/2

Let $\beta \in (0,1)$ and $s > (3+3n)/4$. Let $\mathcal{M}$ be a connected and compact manifold (without boundary!). The unique strong solution to the fractional Cauchy problem

$$\begin{cases} \partial_t^\beta u(m,t) = \Delta u(m,t), \quad m \in \mathcal{M}, t > 0 \\ u(m,0) = f(m), \quad m \in \mathcal{M}, \quad f \in H^s(\mathcal{M}) \end{cases} \tag{31}$$

is given by

$$u(m,t) = \mathbb{E} f(B^m_{E_t}) = \sum_{j=1}^{\infty} E_\beta(-t^\beta \lambda_j)\phi_j(m) \int_{\mathcal{M}} \phi_j(y) f(y) \mu(dy) \tag{32}$$

where B^m_t is a Brownian motion in $\mathcal{M}$ started at $m \in \mathcal{M}$ and $E_t = E^\beta_t$ is inverse to a stable subordinator with index $0 < \beta < 1$.

Subordinated Brownian motion on $\mathcal{M}$

Let B_t^m be a BM on $\mathcal{M}$ started at $m \in \mathcal{M}$ and D_t be a subordinator with symbol ψ. Then, the subordinated process $X(t) = B^m(D_t)$ has infinitesimal generator

$$-\psi(-\Delta) = \int_0^\infty (P_s - I)\,\mu(ds) \tag{33}$$

where P_s is the Brownian semigroup and μ is the Lévy measure corresponding to the symbol ψ.

In particular, let $f \in H^s(\mathcal{M})$ and $s > (3n+3)/4$. Then,

$$-\psi(-\Delta)f(m) = -\sum_{j\in\mathbf{N}} f_j \phi_j(m)\psi(\lambda_j) \tag{34}$$

is absolutely and uniformly convergent ($f_j = \int f\phi_j d\mu$)

What about $B(E_t)$?

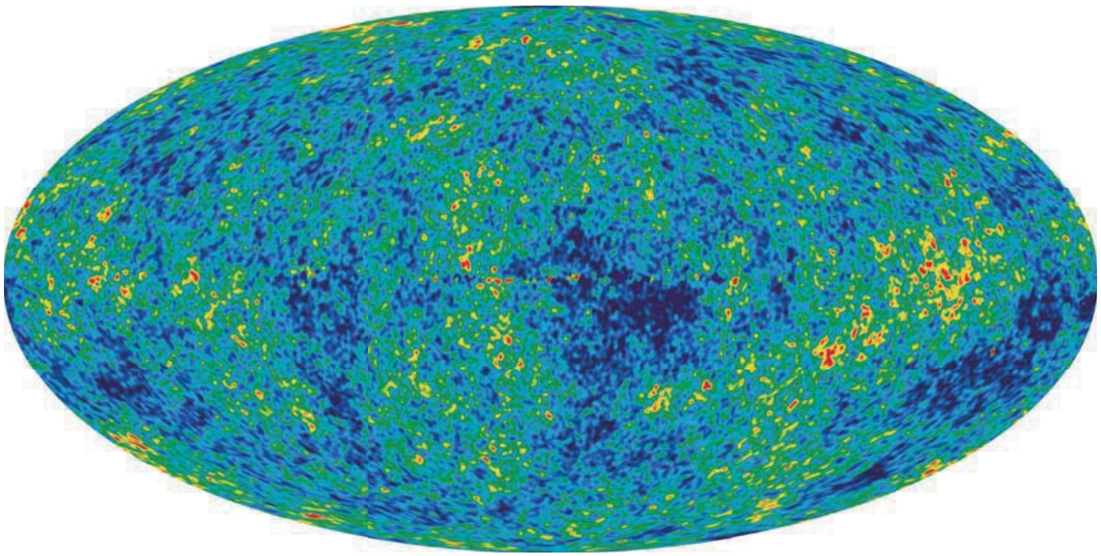
- fractional Cauchy problem
- memory kernel
- slow time and slow "diffusion"

What about $B(D_t)$?

- subordinate problem
- subordinate semigroup
- jumps

Random fields on the sphere

Cosmic microwave background analysis: thermal radiation filling the observable universe almost uniformly.



The Big Bang theory predicts that the initial conditions for the universe are originally random in nature, and inhomogeneities follow a roughly Gaussian probability distribution. The resulting standard model of the Big Bang uses a Gaussian random field

Subordinated rotational Brownian motion

First we recall that

$$\triangle_{\mathbb{S}^2_r} = \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) \tag{35}$$

is the Laplace operator on the sphere $\mathbb{S}^2_1 = \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| = 1\}$ for which

$$\triangle_{\mathbb{S}^2_1} Y_{lm} = -\lambda_l\, Y_{lm} \tag{36}$$

with $\lambda_l = l(l+1)$, $l \geq 0$. For $\theta \in (0,\pi)$, $\varphi \in (0, 2\pi)$

$$Y_{lm}(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P_{lm}(\cos\theta)e^{im\varphi}, \quad |m| \leq l \tag{37}$$

are the spherical harmonics and

$$P_{lm}(z) = (-1)^m (1 - z^2)^{m/2} \frac{d^m}{dz^m} P_l(z) \tag{38}$$

are the associated Legendre functions and P_l are the Legendre polynomials defined by the Rodrigues' formula

$$P_l(z) = \frac{1}{2^l l!} \frac{d^l}{dz^l} (z^2 - 1)^l. \tag{39}$$

The Legendre polynomials $P_l(z)$, $l \geq 0$, satisfy the differential equation

$$\frac{d}{dz}\left((1 - z^2) \frac{d}{dz} P_l(z)\right) = -\lambda_l \, P_l(z) \tag{40}$$

$$\lambda_l = l(l+1), \quad l \geq 0.$$

The unique strong solution to the fractional Cauchy problem on $\mathcal{M} = \mathbb{S}^2_1$ with initial datum $u_0 = f$ is given by

$$u_\alpha(x,t;x_0,t_0) = \mathbb{E} f\left(B^x(E^\alpha_{t-t_0}) - x_0\right) \tag{41}$$

Moreover,

$$u_\alpha(x,t;x_0,t_0) = \sum_{l\geq 0} \frac{2l+1}{4\pi} f_l \, E_\alpha\left(-\lambda_l\,(t-t_0)^\alpha\right) P_l(\langle x, x_0\rangle) \tag{42}$$

where P_l, $l \geq 0$ are the Legendre polynomilas and (Mittag-Leffler)

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad z \in \mathbb{C},\ \Re\{\alpha\} > 0$$

Random Field on the unit sphere

Let us consider the n-weakly isotropic random field $\{T(x); x \in \mathbb{S}_1^2\}$ for which

$$\mathbb{E}T(gx) = 0,\ \mathbb{E}|T(gx)|^n < \infty,\ T(gx) \stackrel{d}{=} T(x),\ \ \text{for all } g \in SO(3).$$

The following spectral representation

$$T(x) = \sum_{l \geq 0} \sum_{m=-l}^{+l} a_{lm} Y_{lm}(x) = \sum_{l \geq 0} T_l(x) \tag{43}$$

holds true in $L^2(dP \otimes d\mu)$, where

$$a_{lm} = \int_{\mathbb{S}_1^2} T(x) Y_{lm}^*(x) \mu(x), \quad l \geq 0,\ |m| \leq l \tag{44}$$

and μ is the Lebesgue measure, $\mu(\mathbb{S}_1^2) = 4\pi$.

We have that

$$\mathbb{E}[T(x)] = 0$$

$$\mathbb{E}[T(x)]^2 = \sum_{l \geq 0} \frac{2l+1}{4\pi} C_l < \infty \qquad \Rightarrow C_l \approx l^{-\gamma}, \gamma > 2$$

$$\mathbb{E}[T_l(x)T_l(y)] = \frac{2l+1}{4\pi} P_l(\langle x, y \rangle) C_l$$

and

$$\mathbb{E}|a_{lm}|^2 = C_l, \quad l \geq 0$$

is the angular power spectrum.

→If T is isotropic, then the harmonic coefficients a_{lm} are zero-mean and uncorrelated over l and m.

→If T is Gaussian, then a_{lm} are Gaussian and independent random coefficients

High-frequency / High-resolution analysis

Since

$$L^2(\mathbb{S}^2_1) = \bigoplus_{l \geq 0} \mathcal{H}_l \tag{45}$$

the random field $T_l(x)$ in

$$T(x) = \sum_{l \geq 0} \sum_{|m| \leq l} a_{lm} Y_{lm}(x) = \sum_{l \geq 0} T_l(x) \tag{46}$$

represents the projection of T on the eigenspace $\mathcal{H}_l$. As l increases we get more and more information about T (about observation in real data). In particular, we are interested on the high-frequency behavior of the angular power spectrum

$$C_l, \quad l > L, \quad L \text{ large} \tag{47}$$

Long range dependence

Let $T(x)$ be the random field previously introduced and X_t a process on $\mathbb{S}^2_1$. We define a coordinates changed time dependent random field as follows

$$\mathfrak{T}_t(x) = T(x + X_t), \quad x \in \mathbb{S}^2_1,\ t > 0. \tag{48}$$

We say that the zero-mean process $\mathfrak{T}_t(x)$ exhibits long range dependence if

$$\sum_{\tau=1}^{\infty} \mathbb{E}[\mathfrak{T}_{t+\tau}(x)\,\mathfrak{T}_t(x)] = \infty. \tag{49}$$

Otherwise, we have short range dependence.

non-Markovian coordinates change

Let $E_t = E_t^\beta$ be an inverse to a stable subordinator of order β and B a BM on $\mathbb{S}^2_1$. For

$$\mathfrak{T}_t^\beta(x) = T(B^x_{E_t}) = T(x + B^{x_N}_{E_t}), \quad x \in \mathbb{S}^2_1,\ t > 0,\ \beta \in (0,1) \tag{50}$$

(x_N is the North Pole) we get that, for $l > L$, L large

$$\sum_{\tau=1}^{\infty} \mathbb{E}[\mathfrak{T}^\beta_{t+\tau,l}(x)\, \mathfrak{T}^\beta_{t,l}(x)] = \infty \tag{51}$$

and

$$C_l^\beta \approx l^{-\gamma} E_\beta(-l^2) \approx \frac{l^{-\gamma-2}}{\Gamma(1-\beta)} - \frac{l^{-\gamma-4}}{\Gamma(1-2\beta)} + \dots, \quad \gamma > 2 \tag{52}$$

Markovian coordinates change

Let D_t be a subordinator with symbol ψ and B a BM on $\mathbb{S}^2_1$. For

$$\mathfrak{T}^\psi_t(x) = T(B^x_{D_t}) = T(x + B^{x_N}_{D_t}), \quad x \in \mathbb{S}^2_1,\ t > 0 \tag{53}$$

(x_N is the North Pole) we get that, for $l > L$, L large

$$\sum_{\tau=1}^{\infty} \mathbb{E}[\mathfrak{T}^\psi_{t+\tau,l}(x)\, \mathfrak{T}^\psi_{t,l}(x)] = \frac{2l+1}{4\pi} \frac{C_l}{e^{\psi(\lambda_l)} - 1} \approx \frac{2l+1}{4\pi} l^{-\gamma} e^{-\psi(l^2)} < \infty \tag{54}$$

and

$$C^\psi_l \approx l^{-\gamma} e^{-\psi(l^2)}, \quad \gamma > 2 \tag{55}$$

Let us consider the sum of an α-stable subordinator, say X_t and, a geometric β-stable subordinator, say Y_t. We assume that X_t and Y_t are independent. For $c, d \geq 0$, the subordinator $X_{ct} + Y_{dt}$, $t > 0$ has the symbol

$$\psi(\xi) = c\,\xi^{\alpha} + d\,\ln(1+\xi^{\beta}), \quad \alpha, \beta \in (0,1).$$

The corresponding covariance function of $\mathfrak{T}_t^{\psi}(x)$ leads to

$$\begin{aligned}\sum_{\tau=1}^{\infty} \mathbb{E}[\mathfrak{T}^{\psi}_{t+\tau,l}(x)\mathfrak{T}^{\psi}_{t,l}(x)] =& \frac{2l+1}{4\pi} C_l \sum_{\tau=1}^{\infty} \left(\frac{e^{-c\lambda_l^{\alpha}}}{(1+\lambda_l^{\beta})^d} \right)^{\tau} \\ =& \frac{2l+1}{4\pi} C_l \frac{1}{e^{c\lambda_l^{\alpha}} + \lambda_l^{d\beta} e^{c\lambda_l^{\alpha}} - 1} \\ \approx& \frac{2l+1}{4\pi} l^{-\gamma-2d\beta} e^{-cl^{2\alpha}}, \quad l \text{ large}\end{aligned}$$

$\rightarrow$ "there exist infinite Bernstein functions" $\leftarrow$

The CMB radiation can be associated with the so-called age of recombination, that is an early stage in the development of the universe. CMB radiation confirms the theory explained by the Big Bang model. Due to Einstein cosmological principle (universe looks identical everywhere in space and appears the same in every direction), the CMB radiation can be also considered as an **isotopic** image of the early universe and therefore, from a mathematical point of view, CMB radiation can be viewed as a realization of an isotropic random field on the sphere, say $T(x)$, $x \in \mathbb{S}_1^2$, for which the variance is finite and the mean equals zero. Furthermore, we require Gaussianity. Indeed, from the prevailing models for early Big Bang dynamics, the so-called inflationary scenario, the random fluctuations have to be **Gaussian**.

The angular power spectrum of the random fields considered in this work exhibits polynomial and/or exponential behavior in the high-frequency (or resolution) analysis and therefore, we introduce a large class of models in which many aspects can be captured, such as Sachs-Wolfe effect (the predominant source of fluctuations, pol decay) or Silk damping effect (also called collisionless damping: anisotropies reduced, universe and CMB radiation more uniform, exp decay). We provide a probabilistic interpretation of the anisotropies of the CMB radiation and we characterize the class $\mathfrak{D}$ introduced in Marinucci-Peccati 2010 by means of the coordinates change of random fields

Thank You!