LAN property for fractional SDEs

Yanghui Liu

Purdue University

AMS sectional meeting,
University of Michigan, October 20-21, 2018.

A joint work with Eulalia Nualart and Samy Tindel.

Outline

Presentation Outline

Introduction

Consider $\mathbb{R}^d$-valued equation:

$$Y_t = y_0 + \int_0^t b(Y_s;\theta)ds + \sigma B_t, \quad t \in [0,\tau].$$

- $y_0 \in \mathbb{R}^d$: given initial.
- $B = (B^1, \ldots, B^d)$: driving process.
- $\theta \in \mathbb{R}^q$: *unknown* parameter.
- $\{b(\cdot;\theta), \theta \in \Theta\}$: known family of drift coefficients. $\Theta \subset\subset \mathbb{R}^q$.
- $\sigma \in \mathbb{R}^{d\times d}$: known diffusion coefficient.

$\hookrightarrow$ We are concerned with the estimation problem of θ while Y is observed.

Fractional Brownian motion

Definition

A fractional Brownian motion $B = (B^1, \ldots, B^d)$ with Hurst parameter $H \in (0, 1)$:

- centered Gaussian process
- $\mathbb{E}(B^i_s B^i_t) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})$
- $\mathbb{E}(B^i_s B^j_t) = 0$ for $i \neq j$

Motivations: finance, geophysics (water cycle, atmosphere, and so on), ethernet traffic, some inverse problems in mathematical biology, and so on.

Parameter estimation results

Available parameter estimation results:

- $b(x;\theta)$ linear in both x and θ: Belfadli, Es-Sebaiy, Ouknine (2011); Hu, Nualart (2010); Kleptsyna, Le Breton (2002); Le Breton (1998), etc.
- $b(x;\theta)$ linear and discrete observation: Brouste, Lacus (2013); Xiao, Zhang, Xu (2011); etc.
- $b(x;\theta)$ linear in θ: Kozachenko, Melnikov, Mishura (2004); Tudor, Viens (2007); Hu, Nualart, Zhou (2018); etc.

Question: parameter estimation for a general drift $b(x;\theta)$ and for multidimensional case?

Presentation Outline

LAN property

Denote P_θ (P_θ^τ): probability law of the solution y in the space $C^\lambda(\mathbb{R}_+; \mathbb{R}^d)$ ($C^\lambda([0,\tau]; \mathbb{R}^d)$).

Definition

The parametric statistical model $\{P_\theta, \theta \in \Theta\}$ satisfies LAN property at $\theta \in \Theta$ if there exist:

- A $q \times q$ invertible matrix $\varphi_\tau(\theta)$,
- A $q \times q$ positive definite matrix $\Sigma(\theta)$,

such that for any $u \in \mathbb{R}^q$,

$$\log\left(\frac{dP^\tau_{\theta+\varphi_\tau(\theta)u}}{dP^\tau_\theta}\right) \xrightarrow{\mathcal{L}} u^T \mathcal{N}(0, \Sigma(\theta)) - \frac{1}{2} u^T \Sigma(\theta) u \qquad \text{as } \tau \to \infty.$$

Minimax theorem

Theorem

Suppose

- $(P_\theta)_{\theta\in\Theta}$ *satisfies the LAN property at some* θ.
- $(\hat{\theta}_\tau)_{\tau\geq 0}$: *a family of estimators of the parameter* θ.

Then for any loss function ℓ *we have:*

$$\lim_{\delta\to 0}\liminf_{\tau\to\infty}\sup_{\theta'\in\Theta:|\theta'-\theta|<\delta}\mathbb{E}_{\theta'}\left[\ell\left(\varphi_\tau^{-1}(\theta)(\hat{\theta}_\tau-\theta')\right)\right]\geq\mathbb{E}[\ell(Z)],$$

where $Z\sim\mathcal{N}(0,\Sigma(\theta)^{-1})$.

Example of a loss function: $\ell(u)=|u|^p$, $p>0$.

Stability assumption

Hypothesis

- regularity assumptions on the drift coefficient b.
- linear growth and inward assumptions on b.
- diffusion coefficient σ invertible.

Ergodic properties

Theorem

Suppose the hypothesis holds.
Then

(*i*) *There exists an ergodic process* $\bar{Y}$ *such that*

$$\lim_{t\to\infty} |Y_t - \bar{Y}_t| = 0, \quad a.s.$$

(*ii*) *For a Frechet differentiable functional* F *on* $C^\beta([-\rho, 0])$, $\beta < H$, *we have:*

$$\lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau F(Y|_{[t-\rho,t]})dt = \mathbb{E}[F(\bar{Y}|_{[-\rho,0]})], \qquad a.s.$$

A Girsanov-type lemma

Lemma

Suppose

- B: *a fractional Brownian motion with* $H > 1/2$
- g: *a process in* $I_+^{H-1/2}(L^2(\mathbb{R}))$
- $Q_t = B_t + \int_0^t g_s ds$
- $D_+^{H-1/2} g$ *satisfies a Novikov-type condition*
- W: *the Wiener process such that* $B_t = \int_{\mathbb{R}_-} (-r)^{H-1/2} [dW_{t+r} - dW_r]$.
- $\tilde{P}$: *the probability defined by* $\left.\frac{d\tilde{P}}{dP}\right|_{[0,\tau]} = e^{-L}$, *where*

$$L = \frac{1}{c(H)} \int_0^\tau [D_+^{H-1/2} g]_u dW_u + \frac{1}{2c(H)^2} \int_0^\tau [D_+^{H-1/2} g]_u^2 du$$

- $c(H)$: *a constant dependent of* H.

Then Q *is a fractional Brownian motion on* $[0, \tau]$ *under* $\tilde{P}$.

A first step

Take $\theta_\tau = \theta + \tau^{-1/2}u$. Then by our Girsanov-type lemma:

$$\begin{aligned}
&\log\left(\frac{dP^\tau_{\theta_\tau}}{dP^\tau_\theta}\right) \\
&= -\frac{1}{c(H)}\int_0^\tau \left\langle \sigma^{-1}\left(\left[D_+^{H-1/2}b(Y;\theta)\right]_t - \left[D_+^{H-1/2}b(Y;\theta_\tau)\right]_t\right), dW_t\right\rangle \\
&\quad - \frac{1}{2c(H)^2}\int_0^\tau \left|\sigma^{-1}\left(\left[D_+^{H-1/2}b(Y;\theta)\right]_t - \left[D_+^{H-1/2}b(Y;\theta_\tau)\right]_t\right)\right|^2 dt \\
&\approx I(\tau) + J(\tau)\,,
\end{aligned}$$

where

$$\begin{aligned}
I(\tau) &= -\frac{1}{2c(H)^2\tau}\int_0^\tau |\sigma^{-1}[D_+^{H-1/2}\hat{b}(Y;\theta)]_t u|^2 dt\,, \\
J(\tau) &= \frac{1}{c(H)\tau^{1/2}}\int_0^\tau \left\langle \sigma^{-1}[D_+^{H-1/2}\hat{b}(Y;\theta)]_t u, dW_t\right\rangle
\end{aligned}$$

and $\hat{b} = \partial_\theta b$.

Convergence of $I(\tau)$

Lemma

Denote

$$\Sigma(\theta) = \tilde{c}_H \int_{\mathbb{R}_+^2} \frac{\mathbb{E}_\theta \left[\left(\hat{b}(\bar{Y}_0;\theta) - \hat{b}(\bar{Y}_{r_1};\theta) \right)^T (\sigma^{-1})^T \sigma^{-1} \left(\hat{b}(\bar{Y}_0;\theta) - \hat{b}(\bar{Y}_{r_1};\theta) \right) \right]}{r_1^{1/2+H} r_2^{1/2+H}} dr_1 dr_2$$

Then $\Sigma(\theta)$ *is integrable, and we have*

$$I \to -\frac{1}{2} u^T \Sigma(\theta) u, \qquad \tau \to \infty.$$

ingredients of proof: fractional calculus, ergodic theorem.

Convergence of $J(\tau)$

By (1) the CLT for Brownian martingales; and (2) the convergence of $I(\tau)$, we obtain:

Lemma

$$J(\tau) \xrightarrow{\mathcal{L}} u^T \mathcal{N}(0, \Sigma(\theta)), \qquad \tau \to \infty.$$

Main result

Putting together the convergences of $I(\tau)$ and $J(\tau)$:

Theorem

Assume the ergodicity property hypothesis holds. Then for any $\theta \in \Theta$ and $u \in \mathbb{R}^q$ fixed, we have

$$\log \frac{dP^{\tau}_{\theta+\frac{u}{\sqrt{\tau}}}}{dP^{\tau}_{\theta}} \xrightarrow{\mathcal{L}} u^T \mathcal{N}(0, \Sigma(\theta)) - \frac{1}{2} u^T \Sigma(\theta) u, \qquad \tau \to \infty.$$

Presentation Outline

Ornstein-Uhlenbeck case

Consider

$$Y_t = -\theta \int_0^t Y_s ds + B_t, \quad t \in [0, \tau], \quad \theta > 0.$$

The stationary solution is given by

$$\bar{Y}_t = \int_{-\infty}^t e^{-\theta(t-s)} dB_s.$$

LAN property - Ornstein-Uhlenbeck process

Proposition

Let Y be the fractional Ornstein-Uhlenbeck process solution. Then the LAN property is satisfied with

$$\Sigma(\theta) = \frac{1}{2\theta}.$$

Remark: in this case $\Sigma(\theta)$ does not depend on the Hurst parameter H.

Efficiency of MLE

Facts: The MLE of a fractional Ornstein-Uhlenbeck process is:
(1) uniformly consistent; (2) uniformly asymptotically normal.

Proposition

The MLE is asymptotically minimax efficient for any loss function $\ell(u) = |u|^p$, $p > 0$.

Thank you very much for your attention!