Testing for Change in Long Memory Stochastic Volatility Time Series

Rafał Kulik

Department of Mathematics and Statistics (University of Ottawa)
Joint work with Annika Betken (Bochum)

Ann Arbor, 20/10/2018

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Long Memory Stochastic Volatility

Let $\{\eta_j, j \in \mathbb{Z}\}$ be a sequence of i.i.d. standard normal random variables. Define

$$Y_j = \sum_{k=1}^{\infty} a_k \eta_{j-k} \ . \tag{1}$$

The coefficients a_k are such that

$$\mathrm{var}\left(\sum_{j=1}^{n} Y_j\right) \sim n^{2-D}\ell^2(n) \ , \quad D \in (0,1) \ .$$

Note: most of the limit theorems in case of long memory are for $f(Y_j)$, where Y_j is as in (1) (with possibly arbitrary marginal distribution).

Long Memory Stochastic Volatility

Let $\{\varepsilon_j, j \geq 0\}$ be a sequence of i.i.d. random variables. For $\phi \geq 0$ define

$$X_j = \phi(Y_j)\varepsilon_j \;, \quad j \geq 0 \;, \quad \mathcal{F}_j = \sigma(\{\eta_i, \varepsilon_i\}, i \leq j) \;. \tag{2}$$

Properties:

- LMSV model: where $\{\eta_j\}$ and $\{\varepsilon_j\}$ are independent.
- Model with leverage: where $\{(\eta_j, \varepsilon_j)\}$ is a sequence of i.i.d. random vectors. For fixed j, ε_j and Y_j are independent, but Y_j may not be independent of the past $\{\varepsilon_i, i < j\}$.

Also, in case of no-leverage:

$$\mathrm{cov}(X_0, X_k) = (\mathbb{E}[\varepsilon_0])^2 \mathrm{cov}(\phi(Y_0), \phi(Y_k)) \;, \quad k \geq 1 \;.$$

In this talk LMSV means a model with leverage.

Some tools for long memory

For a stationary, long range dependent Gaussian process $\{Y_j, j \geq 1\}$ and a measurable function g such that $\mathbb{E}\left[g^2(Y_1)\right] < \infty$ the corresponding Hermite expansion is defined by

$$g(Y_1) - \mathbb{E}\left[g(Y_1)\right] = \sum_{q=m}^{\infty} \frac{J_q(g)}{q!} H_q(Y_1) \,,$$

where H_q is the q-th Hermite polynomial,

$$J_q(g) = \mathbb{E}\left[g(Y_1) H_q(Y_1)\right]$$

and

$$m = \inf\left\{q \geq 1 \,|\, J_q(g) \neq 0\right\} \,. \tag{3}$$

The integer m is called the Hermite rank of g and we refer to $J_q(g)$ as the q-th Hermite coefficient of g.

Some tools for long memory

The asymptotic behavior of partial sums of subordinated Gaussian sequences is characterized in Taqqu (1979). Due to the functional non-central limit theorem in that paper, if $mD < 1$, then

$$\frac{1}{d_{n,m}} \sum_{j=1}^{\lfloor nt \rfloor} g(Y_j) \Rightarrow \frac{J_m(g)}{r!} Z_{m,H}(t), \ 0 \le t \le 1, \tag{4}$$

where $Z_{m,H}(t)$, $0 \le t \le 1$, is an m-th order Hermite process,

$$d_{n,m}^2 = \mathrm{var}\left(\sum_{j=1}^{n} H_m(Y_j)\right) \sim c_m n^{2-mD} L^m(n), \ c_m = \frac{2m!}{(1-Dm)(2-Dm)},$$

and the convergence holds in $\mathbb{D}([0,1])$, See Dehling and Taqqu (1981).

Hypothesis testing

Given the observations $X_1, \dots, X_n$ and a function ψ, we define $\xi_j = \psi(X_j)$, $j = 1, \dots, n$, and we consider the testing problem:

$$\begin{aligned} H_0 :& \mathbb{E}[\xi_1] = \cdots = \mathbb{E}[\xi_n] , \\ H_1 :& \exists\ k \in \{1, \dots, n-1\} \quad \text{such that} \\ & \quad \mathbb{E}[\xi_k] = \cdots = \mathbb{E}[\xi_k] \neq \mathbb{E}[\xi_{k+1}] = \cdots = \mathbb{E}[\xi_n] . \end{aligned}$$

We choose ψ according to the specific change-point problem considered. Possible choices include:

- $\psi(x) = x$ in order to detect changes in the mean of the observations $X_1, \dots, X_n$ (change in location);
- $\psi(x) = x^2$ in order to detect changes in the variance of the observations $X_1, \dots, X_n$ (change in volatility).

Test statistics

- The CUSUM test rejects the hypothesis for large values of the test statistic $C_n = \sup_{0 \le \lambda \le 1} C_n(\lambda)$, where

$$C_n(\lambda) = \left| \sum_{j=1}^{\lfloor n\lambda \rfloor} \psi(X_j) - \frac{\lfloor n\lambda \rfloor}{n} \sum_{j=1}^{n} \psi(X_j) \right| . \tag{5}$$

- The Wilcoxon test rejects the hypothesis for large values of the test statistic $W_n = \sup_{0 \le \lambda \le 1} W_n(\lambda)$, where

$$W_n(\lambda) = \left| \sum_{i=1}^{\lfloor n\lambda \rfloor} \sum_{j=\lfloor n\lambda \rfloor + 1}^{n} \left(1_{\{\psi(X_i) \le \psi(X_j)\}} - \frac{1}{2} \right) \right| . \tag{6}$$

Relevant literature

- iid case: Csörgö and Horvath (1997);
- long memory (subordinated Gaussian or linear processes): Giraitis et al. (1996) - Kolmogorov-Smirnov change point in marginal distribution; Horvath and Kokoszka (1997) - CUSUM test for mean; Dehling et al. (2013) - CUSUM and Wilcoxon test; Betken (2016) - CUSUM and Wilcoxon tests; Tewes (2017) - Kolmogorov-Smirnov and Cramer - von Mises change point tests. CUSUM and Wilcoxon tests are affected by long memory; relative efficiency is 1;
- Nothing available in LMSV case!;
- self-normalization in change point testing for short and long memory linear processes: Shao and Zhang (2010); Shao (2011); Dehling et al. (2013); Betken (2016);
- spurious long memory: Berkes at el. (2006); Section 7.9.1 in Beran et al. (2013).

Partial sums behaviour

In order to determine the asymptotic behavior of the CUSUM test statistic we have to consider the partial sum process

$$\sum_{j=1}^{\lfloor nt \rfloor} \left(\psi(X_j) - \mathbb{E}\left[\psi(X_j)\right] \right) , \quad t \in [0,1] .$$

Let

$$\mathcal{F}_j = \sigma\left(\varepsilon_j, \varepsilon_{j-1}, \ldots, \eta_j, \eta_{j-1}, \ldots\right) , \quad \Psi(y) = \mathbb{E}\left[\psi(\phi(y)\varepsilon_1)\right] .$$

Denote by m the Hermite rank of Ψ and by $J_m(\Psi)$ the corresponding Hermite coefficient.

Let $Z_{m,H}(t)$ be the Hermite-Rosenblatt process with the Hurst index H.

Let $B(t)$ be a standard Brownian motion.

Partial sums behaviour

Theorem 1 (Theorem 4.10 in Beran et al. (2013))

Assume that $\{X_j, j \geq 1\}$ *follows the LMSV model. Furthermore, assume that* $\sigma^2 = \mathbb{E}[\psi^2(X_1)] < \infty$.

1. *If* $\mathbb{E}[\psi(X_1) \mid \mathcal{F}_0] \neq 0$ *and* $mD < 1$, *then*

$$\frac{1}{d_{n,m}} \sum_{j=1}^{\lfloor nt \rfloor} (\psi(X_j) - \mathbb{E}[\psi(X_j)]) \Rightarrow \frac{J_m(\Psi)}{m!} Z_{m,H}(t) \ , \quad t \in [0,1] \ ,$$

   in $\mathbb{D}([0,1])$.

2. *If* $\mathbb{E}[\psi(X_1) \mid \mathcal{F}_0] = 0$, *then*

$$\frac{1}{\sqrt{n}} \sum_{j=1}^{\lfloor nt \rfloor} \psi(X_j) \Rightarrow \sigma B(t) \ , \quad t \in [0,1] \ .$$

Asymptotics for CUSUM test

As an immediate consequence of Theorem 1, we obtain the asymptotic distribution of the CUSUM statistic.

Corollary 2 (Betken and Kulik (2018))

Assume that the conditions of Theorem 1 hold.

1. *If* $\mathbb{E}[\psi(X_1) \mid \mathcal{F}_0] \neq 0$ *and* $mD < 1$*, then*

$$\frac{1}{d_{n,m}} \sup_{0\le\lambda\le1} C_n(\lambda) \Rightarrow \frac{|J_m(\Psi)|}{m!} \sup_{0\le t\le1} |Z_{m,H}(t) - tZ_{m,H}(1)| \ . \qquad (7)$$

2. *If* $\mathbb{E}[\psi(X_1) \mid \mathcal{F}_0] = 0$*, then*

$$\frac{1}{\sqrt{n}} \sup_{0\le\lambda\le1} C_n(\lambda) \Rightarrow \sigma \sup_{0\le t\le1} |B(t) - tB(1)| \ .$$

CUSUM test - comments

- It is important to note that the Hermite rank of Ψ does not necessarily correspond to the Hermite rank of the function $\phi(\cdot)$ that appears in the definition of the LMSV model.
- If $\psi(x) = x$ (change in the mean) and $\mathbb{E}[\varepsilon_0] = 0$, then long memory in LMSV does not affect the CUSUM test.
- If $\psi(x) = x^2$ (change in the variance), then long memory in LMSV affects the CUSUM test.
- For long memory linear processes, long memory always affects CUSUM test (see Horvath and Kokoszka (1997), Betken (2016)).

Sequential empirical process

In order to determine the asymptotic distribution of the Wilcoxon test statistic for the LMSV model, we need to establish an analogous result for the stochastic volatility process $\{X_j, j \geq 1\}$, i.e. our preliminary goal is to prove a limit theorem for the two-parameter empirical process

$$G_n(x,t) = \sum_{j=1}^{\lfloor nt \rfloor} \left(1_{\{\psi(X_j) \leq x\}} - F_{\psi(X_1)}(x) \right),$$

where now $F_{\psi(X_1)}$ denotes the distribution function of $\psi(X_1)$ with $X_1 = \phi(Y_1)\varepsilon_1$. To state the weak convergence, we introduce the following notation:

$$\Psi_x(y) = P\left(\psi(y\varepsilon_1) \leq x\right) .$$

Asymptotics for sequential empirical process

Theorem 3 (Betken and Kulik (2018))

Assume that $\{X_j, j \geq 1\}$ *follows the LMSV model. Moreover, assume that* $\Psi_x(y)$ *is differentiable and that*

$$\int \frac{d}{dy}\Psi_x(y)dy < \infty \,. \tag{8}$$

Let m *denote the Hermite rank of the class*

$$\left\{1_{\{\phi(Y_1)\leq x\}} - F_{\phi(Y_1)}(x),\ x \in \mathbb{R}\right\} \,.$$

If $mD < 1$*, then in* $\mathbb{D}\left([-\infty,\infty] \times [0,1]\right)$*:*

$$\frac{1}{d_{n,m}} G_n(x,t) \Rightarrow \frac{J_m(\Psi_x \circ \sigma)}{m!} Z_{m,H}(t)\,,\ x \in [-\infty,\infty]\,,\ t \in [0,1]\,. \tag{9}$$

Asymptotic for Wilcoxon test

Corollary 4

Under the conditions of Theorem 3

$$\frac{1}{nd_{n,m}} \sup_{\lambda\in[0,1]} W_n(\lambda)$$
$$\Rightarrow \left| \int J_m(\Psi_x \circ \phi) dF_{\psi(X_1)}(x) \right| \frac{1}{m!} \sup_{\lambda\in[0,1]} \left| Z_{m,H}(\lambda) - \lambda Z_{m,H}(1) \right| .$$

Wilcoxon test - comments

- Wilcoxon tests for a change in the mean or change in the variance of LMSV models are typically affected by long memory. This is in line with the findings for subordinated Gaussian processes; cf. Dehling et al. (2013).
- For LMSV, CUSUM test for change in the mean is much more efficient than the Wilcoxon test.

Self-normalization

In order to avoid estimation of the normalization and the unknown coefficients in the limit, we consider the self-normalized CUSUM test statistic. For $0 < \tau_1 < \tau_2 < 1$ it is defined by

$$T_n(\tau_1, \tau_2) = \sup_{k \in \{\lfloor n\tau_1 \rfloor, \ldots, \lfloor n\tau_2 \rfloor\}} \left| G_n(k) \right|,$$

where

$$G_n(k) = \frac{\sum_{j=1}^{k} \xi_j - \frac{k}{n} \sum_{j=1}^{n} \xi_j}{\left\{ \frac{1}{n} \sum_{t=1}^{k} S_t^2(1, k) + \frac{1}{n} \sum_{t=k+1}^{n} S_t^2(k+1, n) \right\}^{\frac{1}{2}}}$$

with $S_t(j,k) = \sum_{h=j}^{t} \left(\xi_h - \bar{\xi}_{j,k} \right), \quad \bar{\xi}_{j,k} = \frac{1}{k-j+1} \sum_{t=j}^{k} \xi_t$. The self-normalized CUSUM test rejects the hypothesis for large values of the test statistic $T_n(\tau_1, \tau_2)$.
Similar ideas for Wilcoxon test.

τ	H	n	self-norm. CUSUM $\alpha = 2.5$ $h=0$	self-norm. CUSUM $\alpha = 2.5$ $h=0.5$	self-norm. CUSUM $\alpha = 2.5$ $h=1$	self-norm. CUSUM $\alpha = 4$ $h=0$	self-norm. CUSUM $\alpha = 4$ $h=0.5$	self-norm. CUSUM $\alpha = 4$ $h=1$	self-norm. Wilcoxon $\alpha = 2.5$ $h=0$	self-norm. Wilcoxon $\alpha = 2.5$ $h=0.5$	self-norm. Wilcoxon $\alpha = 2.5$ $h=1$	self-norm. Wilcoxon $\alpha = 4$ $h=0$	self-norm. Wilcoxon $\alpha = 4$ $h=0.5$	self-norm. Wilcoxon $\alpha = 4$ $h=1$
$\tau = 0.25$	0.6	500	0.046	0.181	0.539	0.042	0.688	0.958	0.032	0.879	0.991	0.030	0.995	1.000
		1000	0.049	0.290	0.722	0.044	0.862	0.990	0.032	0.973	1.000	0.030	1.000	1.000
		2000	0.053	0.458	0.875	0.026	0.967	0.999	0.034	0.999	1.000	0.028	1.000	1.000
	0.7	500	0.051	0.204	0.552	0.042	0.697	0.954	0.029	0.680	0.938	0.021	0.960	0.997
		1000	0.050	0.295	0.727	0.046	0.866	0.990	0.032	0.856	0.988	0.027	0.993	1.000
		2000	0.049	0.455	0.868	0.042	0.966	0.998	0.037	0.948	0.999	0.030	0.999	1.000
	0.8	500	0.045	0.226	0.580	0.044	0.720	0.951	0.031	0.424	0.772	0.021	0.815	0.964
		1000	0.042	0.338	0.736	0.040	0.870	0.989	0.033	0.559	0.862	0.024	0.915	0.984
		2000	0.050	0.498	0.881	0.052	0.960	0.998	0.034	0.673	0.938	0.023	0.958	0.998
	0.9	500	0.044	0.329	0.645	0.041	0.760	0.947	0.031	0.309	0.582	0.020	0.640	0.861
		1000	0.051	0.446	0.761	0.042	0.871	0.980	0.039	0.369	0.650	0.034	0.734	0.912
		2000	0.041	0.585	0.869	0.048	0.949	0.996	0.049	0.422	0.719	0.039	0.791	0.947
$\tau = 0.5$	0.6	500		0.384	0.801		0.904	0.990		0.994	1.000		1.000	1.000
		1000		0.564	0.909		0.973	0.998		1.000	1.000		1.000	1.000
		2000		0.744	0.962		0.993	1.000		1.000	1.000		1.000	1.000
	0.7	500		0.401	0.801		0.902	0.989		0.950	1.000		1.000	1.000
		1000		0.565	0.904		0.972	0.998		0.993	1.000		1.000	1.000
		2000		0.744	0.966		0.994	0.999		1.000	1.000		1.000	1.000
	0.8	500		0.424	0.804		0.899	0.990		0.776	0.977		0.987	0.999
		1000		0.589	0.905		0.966	0.997		0.896	0.995		0.998	1.000
		2000		0.761	0.959		0.994	0.999		0.963	0.999		1.000	1.000
	0.9	500		0.527	0.815		0.893	0.982		0.622	0.890		0.912	0.990
		1000		0.650	0.898		0.959	0.997		0.708	0.936		0.956	0.996
		2000		0.781	0.954		0.989	0.999		0.779	0.960		0.976	0.998

Table 1: Rejection rates of the self-normalized CUSUM and the self-normalized Wilcoxon test (with $\tau_1 = 1 - \tau_2 = 0.15$) for LMSV time series (Pareto distributed $\varepsilon_j, j \geq 1$) of length n with Hurst parameter H, tail index α and a shift in the mean of height h after a proportion τ. The calculations are based on 5,000 simulation runs.

Summary

In general, an application of Wilcoxon-type tests reduces the influence of heavy tails in data generating processes on test decisions. As a result, Wilcoxon-based testing procedures yield better results when testing for changes in LMSV time series. In fact, this is reflected by the simulation results for both of the considered situations, i.e. when testing for changes in the mean and when testing for changes in the variance. In both cases, CUSUM-based change-point tests are outperformed by Wilcoxon-based testing procedures. In particular, the simulation results show that CUSUM-based tests are highly unreliable when testing for a change in the variance.

- For light tailed change point in the mean, use CUSUM test;
- In general, use Wilcoxon test!

Questions

- Regularly α-varying noise ε ($\alpha \in (0,1)$) and $\mathrm{E}[\phi^{\alpha+\delta}(Y_1)] < \infty$ - CUSUM test interplay between long memory and tails? No effect of heavy tails on Wilcoxon test?
- $\phi(Y_1)$ regularly varying ($\alpha \in (0,1)$) and lighter noise. Dichotomy in CUSUM test? No idea about Wilcoxon.

Thank you!!!