Noether theorem for random locations

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Existing results for random locations of some processes

Definition

A mapping $L\colon H \times \mathcal{I} \to \mathbb{R} \cup \{\infty\}$ is called an intrinsic location functional, if it satisfies:

- The mapping $L(\cdot, I) : H \to \mathbb{R} \cup \{\infty\}$ is measurable.
- $L(g, I) \in I \cup \{\infty\}$.
- (Shift compatibility) For every $g \in H$, $I \in \mathcal{I}$ and $c \in \mathbb{R}$,

  $$L(g, I) = L(\theta_c g, I - c) + c,$$

  where $I - c$ is the interval I shifted by $-c$, and by convention, $\infty + c = \infty$.
- (Stability under restrictions) For every $g \in H$ and $I_1, I_2 \in \mathcal{I}$, $I_2 \subseteq I_1$, if $L(g, I_1) \in I_2$, then $L(g, I_2) = L(g, I_1)$.
- (Consistency of existence) For every $g \in H$ and $I_1, I_2 \in \mathcal{I}$, $I_2 \subseteq I_1$, if $L(g, I_2) \neq \infty$, then $L(g, I_1) \neq \infty$.

Results of

- Random locations for stationary processes; (Samorodnitsky and Shen, 2013)
- Random locations for processes with stationary increments; (Shen, 2016)
- Processes combining both a scaling symmetry and a stationarity of the increments.(Shen, 2018).

Above processes: exhibiting certain probabilistic symmetries.
Question: unified framework of random locations with probabilistic symmetries.

1 Previous results

2 Basic Setting

3 Main results

Definition

A stochastic process $\{L(I)\}_{I\in\mathcal{I}}$ indexed by compact intervals and taking values in $\bar{R}$ is called an intrinsic random location, if it satisfies the following conditions:

- For every $I \in \mathcal{I}$, $L(I) \in I \cup \{\infty\}$.
- (Stability under restriction) For every $I_1, I_2 \in \mathcal{I}$, $I_2 \subseteq I_1$, if $L(I_1) \in I_2$, then $L(I_1) = L(I_2)$.
- (Consistency of existence) For every $I_1, I_2 \in \mathcal{I}$, $I_2 \subseteq I_1$, if $L(I_2) \neq \infty$, then $L(I_1) \neq \infty$.

- $\varphi = \{\varphi^t\}_{t\in\mathbb{R}}$: a flow satisfies
  1. $\varphi^0 = Id$;
  2. $\varphi^s \circ \varphi^t = \varphi^{s+t}$;
  3. $\varphi(x,t) = \varphi^t(x) \in C^{1,1}(\mathbb{R}\times\mathbb{R})$;
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- φ-stationary: $\varphi^t(L([a,b])) \overset{d}{=} L([\varphi^t(a), \varphi^t(b)])$.

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- φ-stationary: $\varphi^t(L([a,b])) \stackrel{d}{=} L([\varphi^t(a), \varphi^t(b)])$.
- Define a transform $\tau : (\alpha, \beta) \to \mathbb{R}$ by $\varphi^{\tau(x)}(x_0) = x$, and $L'(I) = \tau(L(\tau^{-1}(I)))$ for $I \in \mathcal{I}$.
- L' is a stationary intrinsic random location.
- Compare intrinsic location functionals and intrinsic random locations.

- Partial order given by L:
  $S := \{x \in \mathbb{R} : x = L(I) \text{ for some } I \in \mathcal{I}\}$, and binary relation "$\preceq$" on S, $x \preceq y$ if there exists $I \in \mathcal{I}$, such that $x, y \in I, L(I) = y$.

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- Point process related to L
  1. $l_x := \sup\{y \in S : y < x, x \preceq y\}$, $r_x := \inf\{y \in S : y > x, x \preceq y\}$.
  2. $\mathcal{E}$: collections of $(l_x, x, r_x) \in \mathbb{R}^3$
  3. Point process: $\xi := \sum_{\epsilon_x \in \mathcal{E}} \delta_{\epsilon_x}$
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  4. Control measure of ξ: $\eta(A) := \mathbb{E}(\xi(A))$ for A.
- For stationary intrinsic random location L on an interval (a, b),

$$P(L([a, b]) \in [u, v]) = \eta((-\infty, a) \times (u, v) \times (b, \infty)).$$

Theorem

Let L be a φ-stationary intrinsic random location. Then for any $I = [a,b]$, the distribution of $L(I)$ has a càdlàg density function, denoted by f, which satisfies

$$\dot{\varphi}^0(x_2)f(x_2) - \dot{\varphi}^0(x_1)f(x_1) = \nu_{\varphi}^{(a,b)}((x_1,x_2]) - \mu_{\varphi}^{(a,b)}((x_1,x_2])$$

where $\dot{\varphi}^0(x)$ is the partial derivative of φ with respect to t at time 0, $\mu_{\varphi}^{(a,b)}$ and $\nu_{\varphi}^{(a,b)}$ are the pull-backs of $\mu^{(a,b)}$ and $\nu^{(a,b)}$ under τ.

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- $$\mu^{(a,b)}([w, y)) = \eta((z_1, z_2, z_3) : z_1 \in [a, a+1), z_2 \in [z_1+w-a, z_1+y-a), z_3 \in (z_1+b-a, \infty))$$

- $$\nu^{(a,b)}([w, y)) = \eta((z_1, z_2, z_3) : z_1 \in (-\infty, z_3+a-b), z_2 \in [z_3+w-b, z_3+y-b), z_3 \in (b, b+1])$$

Conservation law

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Corollary

Denote by $f_t(x)$ *the density of* $L([\varphi^t(a_0), \varphi^t(b_0)])$ *at point* x,
$K(y) = \nu_\varphi^{(a_0,b_0)}((x_0, y]) - \mu_\varphi^{(a_0,b_0)}((x_0, y])$ *for* $y \in (a_0, b_0)$.

$$\dot{\varphi}^0(x)f_t(x) - K((\varphi^t)^{-1}(x))$$

is a constant for t *satisfying* $x \in (\varphi^t(a_0), \varphi^t(b_0))$.

Boundary and near boundary behavior

- $\mathbf{X} = \{X(t)\}_{t\geq 0}$ be a continuous semimartingale with stationary increments;
- $\tau_{\mathbf{X},I} := \inf\{t \in I : X(t) = \sup_{s\in I} X(s)\}$ is the location of the path supremum.
- $\tau_{\mathbf{X},I}$ is almost surely unique;
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- $\tau_{\mathbf{X},I}$ is almost surely unique;
- Local martingale part of $\mathbf{X}$ almost surely does not have any flat part;
- $P(\tau_{\mathbf{X},I} = a) = P(\tau_{\mathbf{X},I} = b) = 0$, and the density of τ exploded near a or near b.

Reference

Samorodnitsky, G. and Shen, Y. (2013). Intrinsic location functionals of stationary processes. *Stochastic Processes and their Applications*, 123(11):4040–4064.

Shen, Y. (2016). Random locations, ordered random sets and stationarity. *Stochastic Processes and their Applications*, 126(3):906–929.

Shen, Y. (2018). Location of the path supremum for self-similar processes with stationary increments. *Annales de l'Institut Henri Poincaré (B)*. (to appear).