Parameter estimation for general Gaussian sequences using analysis on Wiener space

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"AMS meeting Ann Arbor, Michigan, October 20-21,2018".

Introduction

In this talk, we will show how analysis on Wiener space namely the combination between Stein's method and Malliavin calculus which was initiated by Nourdin and Peccati in 2008 can be used to solve parameter estimation problems for fBm-driven models and for more general stationary and non-stationary Gaussian models.

Quick review about the drift estimators

There are several approaches to estimate drift parameters in fBm-driven models, which have been developed over the past 15 years.

- The MLE approach : based on Girsanov transforms and depends on the properties of the deterministic fractional operators related to the fBm. MLE is not easily computable. It relies on being able to compute stochastic integrals with respect to fBm which is difficult and Skorohod-type integrals can not be computed based on the data.
- The LS approach relies on an unobservable Skorohod integral.

Motivations

This work continues the line of research of the paper of Es-Sebaiy and Viens [10], where they solved several parameter estimation problems for observations that are assumed to be **stationary** or **asymptotically so**.

One of the applications they did concerns estimating the drift parameter of the FOU, which is the solution of the SDE

$$dX_t = -\theta X_t dt + dB_t^H, \quad t \geq 0, \qquad X_0 = 0. \tag{1}$$

where B^H, is a fBm of Hurst parameter $H \in (0, 1)$.
Equation (1) has the explicit solution

$$X_t = \int_0^t e^{-\theta(t-s)}\, dB_s^H \tag{2}$$

X_t can be written as

$$X_t = Z_t^\theta - e^{-\theta t} Z_0^\theta$$

where $Z_t^\theta = \int_{-\infty}^t e^{-\theta(t-s)} dB_s^H$ is stationary and ergodic.
They work with the following estimator

$$Q_{f_q,n}(Z^\theta) := \frac{1}{n} \sum_{i=0}^{n-1} f_q(Z_i^\theta)$$

and

$$\lambda_{f_q}(Z^\theta) := E\left[f_q(Z_0^\theta)\right].$$

where f_q is a polynomial function such that f_q possesses the following decomposition

$$f_q(x) := \sum_{k=0}^{q/2} d_{f_q,2k} H_{2k}\left(\frac{x}{\sqrt{r_{Z^\theta}(0)}}\right) \tag{3}$$

where q is an even integer,

They proved that :

- $Q_{f_q,n}(X) \to \lambda_{f_q}(Z^\theta)$ a.s. as $n \to +\infty$.
- $\sqrt{n}(Q_{f_q,n}(X) - \lambda_{f_q}(Z^\theta)) \to \mathcal{N}(0, u_{f_q}(Z^\theta))$ in law as $n \to +\infty$.

where $u_{f_q}(Z^\theta) := \lim_{n\to+\infty} E\left[(\sqrt{n}(Q_{f_q,n}(Z^\theta) - \lambda_{f_q}(Z^\theta)))^2\right]$
Then writing $\lambda_{f_q}(Z^\theta) = \mu_{f_q}(\theta)$, we can consider the following estimator (if μ_{f_q} is invertible)

$$\hat{\theta}_{f_q,n} = \mu_{f_q}^{-1}[Q_{f_q,n}(X)]$$

In the case where $q = 2$, $\mu_{f_2}(\theta) = \frac{H\Gamma(2H)}{\theta^{2H}}$ and

$$\hat{\theta}_n = \left(\frac{1}{nH\Gamma(2H)} \sum_{k=1}^{n} X_k^2 \right)^{-\frac{1}{2H}}$$

They proved that

- $\hat{\theta}_n \to \theta$ a.s. as $n \to +\infty$.
- $\sqrt{v_n}(\hat{\theta}_n - \theta) \to \mathcal{N}(0, \sigma_H^2 \theta)$ in law as $n \to +\infty$.

Now when the driving noise in (1) is any Gaussian process how to prove that

- $\hat{\theta}_n$ remains a good estimator ?
- How to calculate the limiting law ?
- And speed of this convergence in law?

without relying on the **ergodicity** or the **stationarity**?

General Context

Let $X = \{X_n = I_1(f_n)\}_{n\geq 0}$ be a sequence of random variables such that $I_1(f_n)$ is a Wiener Integral with respect to a Gaussian process G where $f_n \in \mathfrak{H}$. In particular, **we don't assume that X is stationary**.

We aim to estimate the asymptotic variance of $\{X_n\}_{n\geq 0}$

$$\lim_{n\to+\infty} E[X_n^2] = f\ ?.$$

Based on the following statistic:

$$\hat{f}_n(X) = \frac{1}{n}\sum_{i=1}^{n} X_i^2. \tag{4}$$

we also define

$$A_n(X) := E[\hat{f}_n(X)] \quad \text{and} \quad V_n(X) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(X_i^2 - E[X_i^2]).$$

The following assumptions are required

($\mathcal{H}1$) $E[X_n^2] \longrightarrow f$, when $n \longrightarrow \infty$.

($\mathcal{H}2$) $E[V_n(X)^2] \longrightarrow \sigma^2$, when $n \longrightarrow \infty$.

($\mathcal{H}3$) $|E[X_t X_s]| \leq C\rho(|t-s|)\ \forall t, s \in \mathbb{R}$ where $t \neq s$ where $\rho : \mathbb{R} \to \mathbb{R}$ is a symmetric function such that $\rho(0) = 1$.

($\mathcal{H}4$) $\sqrt{n}|A_n(X) - f| \longrightarrow 0$, as $n \longrightarrow \infty$.

The strong consistency

Theorem

Assume that $(\mathcal{H}1)$ *and* $(\mathcal{H}2)$ *hold . Then*

$$\hat{f}_n(X) \longrightarrow \mathrm{f}$$

almost surely $n \longrightarrow \infty$.

Asymptotic normality

Theorem (see [6])

Fix an integer $q \geq 2$ and consider a sequence of random variables $\{F_n; n \geq 1\}$ belonging to the qth Wiener chaos of an isonormal Gaussian process and such that $E[F_n^2] = 1$ and let $N \sim \mathcal{N}(0,1)$ be a standard Gaussian random variable. Then, there exists $c, C > 0$ such that

$$c \max\{|E[F_n^3]|, E[F_n^4] - 3\} \leq d_{TV}(F_n, N) \leq C \max\{|E[F_n^3]|, E[F_n^4] - 3\}$$

We set $v_n(X) = E[V_n(X)^2]$ and $F_n(X) = \frac{V_n(X)}{\sqrt{v_n}} = I_2(g_n)$ where $g_n = \frac{1}{\sqrt{n v_n(X)}} \sum_{k=1}^n f_k^{\otimes 2}$.

Remark

Since $E[F_n(X)] = 0$, *we have* $\kappa_3(F_n(X)) = E[F_n(X)^3]$, *and* $E[F_n(X)^2] = 1$, $\kappa_4(F_n(X)) = E[F_n(X)^4] - 3$.

Sous $(\mathcal{H}3)$

$$\begin{aligned}
\kappa_3(F_n(X)) &= 8\langle g_n, g_n \otimes_1 g_n\rangle^2_{\mathfrak{H}^{\otimes 2}} \\
&= \frac{8}{(nv_n(X))^{3/2}} \sum_{i,j,k=1}^{n} \langle f_i, f_k\rangle_{\mathfrak{H}} \langle f_i, f_j\rangle_{\mathfrak{H}} \langle f_k, f_j\rangle_{\mathfrak{H}} \\
&= \frac{8}{(nv_n(X))^{3/2}} \sum_{i,j,k=1}^{n} E[X_i X_k] E[X_i X_j] E[X_j X_k] \\
&\leq \frac{8}{(nv_n(X))^{3/2}} \sum_{i,j,k=1}^{n} \rho(i-k)\rho(i-j)\rho(j-k) \\
&\leq \frac{8}{v_n(X)^{3/2}\sqrt{n}} \Big(\sum_{|k|<n} |\rho(k)|^{3/2}\Big)^2. \qquad (5)
\end{aligned}$$

Under $(\mathcal{H}3)$, we have

$$\kappa_4(F_n(X)) = \frac{1}{v_n(X)^2 n^2} \sum_{k_1,k_2,k_3,k_4=1}^{n} E[X_{k_1}X_{k_2}]E[X_{k_2}X_{k_3}]E[X_{k_3}X_{k_4}]E[X_{k_4}X_{k_1}]$$

$$\leq \frac{1}{v_n(X)^2 n^2} \sum_{k_1,k_2,k_3,k_4=1}^{n} \rho(k-l)\rho(i-j)\rho(k-i)\rho(l-j)$$

$$\leq \frac{1}{v_n(X)^2 n} (\sum_{|k|<n} |\rho(k)|^{\frac{4}{3}})^3 \quad (6)$$

Theorem

Let $N \sim \mathcal{N}(0,1)$. *If* $(\mathcal{H}3)$ *holds and if exists* $\frac{1}{2} < \beta < 2$ *such that* $\rho(t) \leq C|t|^{-\beta}$ *for a large* $|t|$. *Then*

$$d_{TV}(F_n(X), N) \leq \frac{C}{v_n(X)^2 \wedge v_n(X)^{3/2}} \begin{cases} 1 & \text{if } \beta = \frac{1}{2} \\ n^{\frac{3}{2}-3\beta} & \text{if } \beta \in \left(\frac{1}{2}, \frac{2}{3}\right) \\ n^{-\frac{1}{2}} \log(n)^2 & \text{if } \beta = \frac{2}{3} \\ n^{-\frac{1}{2}} & \text{if } \beta \in \left(\frac{2}{3}, 2\right) \end{cases}$$

Theorem

Let $N \sim \mathcal{N}(0,1)$*. If* $(\mathcal{H}3)$ *holds and exits* $\frac{1}{2} < \beta < 2$ *such that* $\rho(t) \leq C|t|^{-\beta}$ *for a large* $|t|$*. Then*

$$d_W\left(\sqrt{\frac{n}{v_n(X)}}\left(\hat{f}_n(X) - f\right), N\right) \leq \sqrt{\frac{n}{v_n(X)}}|A_n(X) - f|$$

$$+\frac{C}{v_n(X)^2 \wedge v_n(X)^{3/2}} \begin{cases} 1 & si\ \beta = \frac{1}{2} \\ n^{\frac{3}{2}-3\beta} & si\ \beta \in \left(\frac{1}{2}, \frac{2}{3}\right) \\ n^{-\frac{1}{2}} \log(n)^2 & si\ \beta = \frac{2}{3} \\ n^{-\frac{1}{2}} & si\ \beta \in \left(\frac{2}{3}, 2\right) \end{cases}$$

Moreover if $(\mathcal{H}2)$ holds,

$$d_W\left(\frac{\sqrt{n}}{\sigma}\left(\hat{f}_n(X)-f\right),N\right) \leq C\left(\sqrt{n}|A_n(X)-f|+|v_n(X)-\sigma^2|\right) + C\begin{cases} n^{\frac{3}{2}-3\beta} & \text{if } \beta\in\left(\frac{1}{2},\frac{2}{3}\right) \\ n^{-\frac{1}{2}}\log(n)^2 & \text{if } \beta=\frac{2}{3} \\ n^{-\frac{1}{2}} & \text{if } \beta\in\left(\frac{2}{3},2\right). \end{cases}$$

In particular, if $(\mathcal{H}2)$-$(\mathcal{H}4)$ hold, then

$$\sqrt{n}(\hat{f}_n(X)-f)\xrightarrow{law}\mathcal{N}(0,\sigma^2)$$

when $n\to\infty$.

Application to gaussian process

Consider an Ornstein-Uhlenbeck process $X = \{X_t\}_{t \geq 0}$ which is defined by the following SDE

$$dX_t = -\theta X_t dt + dG_t, \qquad X_0 = 0; \tag{7}$$

where G is a gaussian process holderien with a positif exponent and $\theta > 0$ is an unknow parameter. The solution of (4) is

$$X_t = \int_0^t e^{-\theta(t-s)} \, dG_s, \qquad t \geq 0$$

where the integral is understood in the Young sense.

SUbfractional O.U

Suppose that G in (7) is a SubFBM S^H such that $H \in (0,1)$ i.e a centred gaussian process, with covariance function

$$R_{S^H}(s,t) := E\left(S_t^H S_s^H\right) = t^{2H} + s^{2H} - \frac{1}{2}\left((t+s)^{2H} + |t-s|^{2H}\right).$$

When $H = \frac{1}{2}$, $S^{\frac{1}{2}}$ is a BM. By Kolmogorov's Criteria

$$E\left(S_t^H - S_s^H\right)^2 \leq (2 - 2^{2H-1})|s-t|^{2H};\ s,\ t \geq\ 0,$$

S^H is Hölderien with exponent $H - \varepsilon$, $\forall \varepsilon \in (0, H)$.

Proposition

If G dans (7) is a SubFBM S^H such that $H \in (0,1)$. Then for a large t,

$$|E[X_t^2] - f_H(\theta)| \leq Ct^{2H-2}$$

where $f_H(\theta) = \frac{H\Gamma(2H)}{\theta^{2H}}$ *and*

$$\sqrt{n}|A_n(X) - f_H(\theta)| \leq C \begin{cases} n^{-\frac{1}{2}} & \text{if } \; 0 < H < 1/2 \\ n^{2H-3/2} & \text{if } \; H \geq 1/2. \end{cases}$$

Proposition

For $H \in (0,1)$ *then* $(\mathcal{H}3)$ *holds, and we have* $\forall t, s \in \mathbb{R}$ *such that* $t \neq s$*,*

$$|E[X_t X_s]| \leq C|t-s|^{2H-2}.$$

Proposition

For all $0 < H < 3/4$,

$$|E[V_n(X)^2] - \sigma_H^2| \trianglelefteq \begin{cases} n^{-1} & si \quad 0 < H < 1/2 \\ \\ n^{4H-3} & si \quad 1/2 \leq H < 3/4 \end{cases}$$

where $\sigma_H^2 = 2\sum_{i\in\mathbb{Z}} \rho_H(i)^2$ *with* $\rho_H(k) := E[Z_k^H Z_0^H]$, $k \in \mathbb{N}$, *where* $Z_k^H := \mathrm{e}^{-\theta k} \int_{-\infty}^{k} \mathrm{e}^{\theta u}\, dB_u^H$, *and* $\rho_H(0) = f_H(\theta) = \frac{H\Gamma(2H)}{\theta^{2H}}$. *In particular* $(\mathcal{H}2)$ *holds.*

If $H = 3/4$, *we have*

$$\left| \frac{E[V_n(X)^2]}{\log(n)} - \frac{9}{16\theta^4} \right| \trianglelefteq \log(n)^{-1}.$$

Theorem

Let $0 < H < 1$, then we have

$$\hat{f}_n(X) \longrightarrow f_H(\theta) \tag{8}$$

a.s when $n \longrightarrow \infty$, where $f_H(\theta) = \frac{H\Gamma(2H)}{\theta^{2H}}$.

For the limiting law of $\hat{f}_n(X)$ when $0 < H \leq \frac{3}{4}$. We apply the results of section (1) by taking $\beta = 3 - 2H$.

Theorem

Let $0 < H < 3/4$ and $N \sim \mathcal{N}(0,1)$, then

$$d_W\left(\sqrt{n}(\hat{f}_n(X) - f_H(\theta))/\sigma_H, N\right) \leq C \begin{cases} n^{-\frac{1}{2}} & si \quad 0 < H < 1/2 \\ n^{2H-3/2} & si \quad 1/2 \leq H < 3/4 \end{cases}$$

and if $H = 3/4$, we have

$$d_W\left(\frac{\sqrt{n}(\hat{f}_n(X) - f_H(\theta))}{\sigma_H\sqrt{\log(n)}}, N\right) \leq C\log(n)^{-1/2}.$$

In particular, if $0 < H < 3/4$, the when $n \to \infty$

$$\sqrt{n}(\hat{f}_n(X) - f_H(\theta)) \xrightarrow{law} \mathcal{N}(0, \sigma_H^2)$$

and if $H = 3/4$,

$$\frac{\sqrt{n}(\hat{f}_n(X) - f_H(\theta))}{\sqrt{\log(n)}} \xrightarrow{law} \mathcal{N}(0, \sigma_H^2)$$

as $n \to \infty$.

Bifractional OU

If G in (7) is a BifFBM $B^{H,K}$ of parameters $H \in (0,1)$ and $K \in (0,1]$. $B^{H,K}$ is a Gaussian process, centred with covariance function

$$E(B_s^{H,K} B_t^{H,K}) = \frac{1}{2^K} \left(\left(t^{2H} + s^{2H} \right)^K - |t-s|^{2HK} \right).$$

When $K = 1$, $B^{H,1}$ is a fBM. $B^{H,K}$ verifies

$$E\left(\left| B_t^{H,K} - B_s^{H,K} \right|^2 \right) \leq 2^{1-K} |t-s|^{2HK},$$

so $B^{H,K}$ has $(HK - \varepsilon)-$-Holder continuous paths for any $\varepsilon \in (0, HK)$ thanks to Kolmogorov's continuity criterion.

Proposition

Suppose that $H \in (0,1)$ and that $K \in (0,1]$, when for a large $\forall t > 0$, we have

$$|E[X_t^2] - f_{H,K}(\theta)| \leq Ct^{2HK-2}$$

où $f_{H,K}(\theta) = 2^{1-K} HK\Gamma(2HK)/\theta^{2HK}$.
Hence

$$\sqrt{n}|A_n(X) - \mathrm{f}_{H,K}(\theta)| \leq C \begin{cases} n^{-\frac{1}{2}} & \text{if } 0 < HK < 1/2 \\ n^{2HK-3/2} & \text{if } HK \geq 1/2. \end{cases}$$

In particular when $HK < 3/4$, the assumption $(\mathcal{H}4)$ holds.

Proposition

For all $(H, K) \in (0,1) \times (0,1]$*, the assumption* $(\mathcal{H}3)$ *holds. And we have* $\forall t, s \in \mathbb{R}$ *such that* $t \neq s$*,*

$$|E[X_t X_s]| \leq C|t-s|^{2HK-2}.$$

Proposition

For all $0 < HK < 3/4$,

$$|E[V_n(X)^2] - \sigma^2_{H,K}| \trianglelefteq \begin{cases} n^{-1} & \text{if} \quad 0 < HK < 1/2 \\ \\ n^{4HK-3} & \text{if} \quad 1/2 \leq HK < 3/4 \end{cases}$$

where

$$\sigma^2_{H,K} = 4 \sum_{i \in \mathbb{N}^*} (\rho_{H,K}(i) - (1 - 2^{1-K})\, e^{-\theta i}\, \rho_{H,K}(0))^2 + 2^{3-2K} \rho_{H,K}(0)^2$$

where $\rho_{H,K}(k) := E[Z^H_k Z^H_0]$, $k \in \mathbb{N}$ *and* $\rho_{H,K}(0) = \frac{HK\Gamma(2HK)}{\theta^{2HK}}$. *In particular, the assumption* $(\mathcal{H}2)$ *holds. If* $HK = 3/4$, *we have*

$$\left| \frac{E[V_n^2]}{\log(n)} - \frac{9}{16\theta^4} \right| \trianglelefteq \log(n)^{-1}.$$

Theorem

Let $0 < H < 1$ *and* $0 < K < 1$*, then we have*

$$\hat{f}_n(X) \longrightarrow f_{H,K}(\theta) \tag{9}$$

a.s when $n \longrightarrow \infty$*, where* $f_{H,K}(\theta) = 2^{1-K} HK\Gamma(2HK)/\theta^{2HK}$.

Theorem

Let $0 < HK < 3/4$ *and* $N \sim \mathcal{N}(0,1)$, *then*

$$d_W\left(\sqrt{n}(\hat{f}_n(X) - f_{H,K}(\theta))/\sigma_{H,K}, N\right) \leq C \begin{cases} n^{-\frac{1}{2}} & \text{if } \; 0 < HK < 1/2 \\ \\ n^{2H-3/2} & \text{if } \; 1/2 \leq HK < 3/4 \end{cases}$$

and if $HK = 3/4$, *we have*

$$d_W\left(\frac{\sqrt{n}(\hat{f}_n(X) - f_{H,K}(\theta))}{\sigma_{H,K}\sqrt{\log(n)}}, N\right) \leq C \log(n)^{-1/2}.$$

In particular, if $0 < HK < 3/4$, then when $n \to \infty$

$$\sqrt{n}(\hat{f}_n(X) - f_{H,K}(\theta)) \xrightarrow{law} \mathcal{N}(0, \sigma^2_{H,K})$$

and if $HK = 3/4$,

$$\frac{\sqrt{n}(\hat{f}_n(X) - f_{H,K}(\theta))}{\sqrt{\log(n)}} \xrightarrow{law} \mathcal{N}(0, \sigma^2_{H,K})$$

as $n \to \infty$.

Thank you for your attention

Biermé, H., Bonami, A., Nourdin, I. and Peccati, G. (2012). Optimal Berry-Esséen rates on the Wiener space: the barrier of third and fourth cumulants. *ALEA* **9**, no. 2, 473-500.

Bercu, B., Nourdin, I. and Taqqu,M.(2010).Almost sure central limit theorem on the Wiener space, *SPA* **120**.1607-1628.

El Machkouri, M., Es-Sebaiy, K. and Ouknine, Y. (2016). Least squares estimator for non-ergodic Ornstein-Uhlenbeck processes driven by Gaussian processes. *Journal of the Korean Statistical Society* **45**, 329–341.

Hu, Y. and Nualart, D. (2010). Parameter estimation for fractional Ornstein-Uhlenbeck processes. *Statist. Probab. Lett.* **80**, 1030-1038.

Hu, Y. and Song, J. (2013). Parameter estimation for fractional Ornstein-Uhlenbeck processes with discrete observations. *F. Viens et al (eds), Malliavin Calculus and Stochastic Analysis: A Festschrift in Honor of David Nualart, 427-442, Springer.*

- Nourdin, I. and Peccati, G. (2015). The optimal fourth moment theorem. *Proc. Amer. Math. Soc.* **143**, 3123-3133.
- Nourdin, I. and Peccati, G. (2012). Normal approximations with Malliavin calculus : from Stein's method to universality. *Cambridge Tracts in Mathematics 192. Cambridge University Press, Cambridge.*
- Nualart, D. (2006). The Malliavin calculus and related topics. *Springer-Verlag, Berlin.*
- Nualart, D. and Peccati, G. (2005). Central limit theorems for sequences of multiple stochastic integrals. *Ann. Probab.* **33**, no. 1, 177-193.
- Es-Sebaiy, K. and Viens, F. (2018). Optimal rates for parameter estimation of stationary Gaussian processes, SPA.
- Barboza L.A and Viens, F. (2017) Parameter estimation of Gaussian stationary processes using the generalized method of moments. *Electron. J. Statist.* **11**, no 1, 401-439.

Tudor, C. and Viens, F. (2007). Statistical aspects of the fractional stochastic calculus. *Ann. Statist.* **35**, no. 3, 1183-1212.

Young, L. C. (1936). An inequality of the Hölder type connected with Stieltjes integration. *Acta Mathematica*, 251-282.