A non-central limit theorem on heavy-tailed chaos.

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Limit theorems for sums

Classical problem:

$\{X(n)\}$: (centered) stationary sequence.
Goal: establish limit theorem of the form:

$$\frac{1}{c(N)} \sum_{1 \le n \le Nt} X(n) \Rightarrow Z(t), \quad \text{as } N \to \infty.$$

• Standard case:
light tails (finite variance) + weak dependence (summable covariance)

$$c(N) \sim N^{1/2}, \quad Z(t): \text{ Brownian motion.}$$

• Nonstandard cases:

- Heavy tails only: e.g., $P(|X(n)| > x) \sim x^{-\alpha}$, $\alpha \in (0,2)$,

$$c(N) \sim N^{1/\alpha}, \quad Z(t): \text{symmetric } \alpha\text{-stable motion}$$

- Long memory only: $\mathrm{Cov}(X(n), X(0)) \sim n^{\beta-1}$, $\beta \in (0,1)$,

$$c(N) \sim N^{1/2+\beta/2}, \quad Z(t): \text{ fractional Brownian motion, ...}$$

• Heavy tails + Long memory?

A model with heavy tails + long memory

- Owada & Samorodnitsky (2015):

$$X(n) = \int_E f(T^n x) M(dx).$$

  - $(E, \mathcal{E}, \mu)$: a measure space with $\mu(E) = \infty$ (infinite-ergodic theory);
  - $T : E \to E$, μ-preserving, ergodic and

    conservative : $\displaystyle\sum_{n=1}^{\infty} 1_A(T^n x) = \infty$ for a.e. $x \in E \;\; \forall A \in \mathcal{E}, \;\; \mu(A) > 0.$

    Memory generated by "recurrence".
  - $f : E \to \mathbb{R}$, supported on "small" set $A \subset E$, $\mu(A) \in (0, \infty)$.
  - M : heavy-tailed random measure with control measure μ:

$$\mathbb{E} e^{i\theta M(A)} = \exp\left(-\mu(A) \int_{\mathbb{R}} (1 - \cos(\theta y)) \rho(dy)\right),$$

$$\rho : \text{symmetric Lévy measure, } \rho((x, \infty)) \sim x^{-\alpha}$$

- Contrast: long-memory moving average

$$X(n) = \int_{\mathbb{Z}} f(x+n) M(dx); \text{ shift } T(x) = x + 1 \textbf{ dissipative}, \quad f\text{: power-law decay;}$$

  - *Memory generated by "large" support of f.*

Limit theorem of Owada & Samorodnitsky (2015)

Recall: $X(n) = \int_E f(T^n x) M(dx)$, $\mathrm{supp}(f) \subset A$. Let $\beta \in (0,1)$.

Under additional conditions, include one implying (Darling-Kac)
If random element $X \sim P_0$ $\forall P_0 \ll \mu$, then successive visits of $T^n X$ to A behaves like renewals whose inter-arrival tail distribution decays like $x^{-\beta}$.
In particular, $\sum_{n=1}^N 1_A(T^n X)$ scales to inverse β-stable subordinator.

This type of behavior only possible when $\mu(E) = \infty$, where $\{T^n\}$ can be **null**.

- Limit theorem obtained with normalization $c(N) \sim N^{\beta(1-1/\alpha)+1/\alpha}$ and limit

$$Z(t) = \mu(f) \int_{\Omega' \times [0,\infty)} L_\beta((t-v)_+, \omega') S_\alpha(d\omega', dv)$$

  - $(\Omega', \mathcal{F}', P')$: a probability space;
  - $S_\alpha(d\omega', dv)$: SαS random measure with control measure $P' \times v^{-\beta} dv$;
  - $L_\beta(t, \cdot)$: Inverse β-stable subordinator (β-Mittag-Leffler process) under P';
  - "Improper" law $v^{-\beta} dv$ makes $M_\beta((t-v)_+, \cdot)$ increment-stationary.
- Contrast: for long-memory moving average $\int_{\mathbb{R}} f(x+n) M(dx)$, $f(x) \sim x^{\beta/2-1}$,

$$c(N) \sim N^{\beta/2+1/\alpha}, \quad Z(t) : \text{linear fractional S}\alpha\text{S motion.}$$

Heavy tails + long memory + nonlinearity

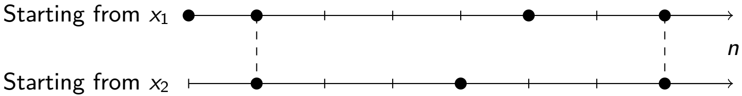
Our goal: consider the following nonlinear extension of $\int_E f(T^n x)M(dx)$:

$$X(n) = \int'_{E\times E} f(T^n x_1, T^n x_2)M(dx_1)M(dx_2),$$

- $f : E \times E \to \mathbb{R}$ supported on $A \times A$;
- Prime $'$ indicates exclusion of diagonal $x_1 = x_2$ in the double integral;
  - If $'$ absent, then diagonal $\int_E f(T^n x, T^n x)M(dx)^2$ will dominate;
  - $X(n)$ needs no centering.

• A first understanding:

Let for simplicity $f(x_1, x_2) = 1_{A\times A}(x_1, x_2) = 1_A(x_1) \times 1_A(x_2)$.



"•" marks a visit of $\{T^n x_i\}$ to A.

- To retain conservativity of $\{(T \times T)^n\}$, need an assumption:

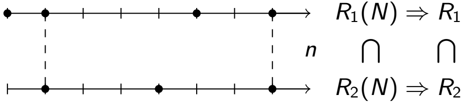
$$\beta > 1/2.$$

- Analog: non-central limit theorem in Gaussian subordination.

Renewal heuristics

$Y_i(n)$: i.i.d. β-heavy-tailed $\mathbb{Z}_+$-valued renewal processes (positive random walk).
Define random set $R_i(N) = \{Y_i(n)/N : n \in \mathbb{Z}_+\}$, scaled renewal epochs.

- $R_i(N) \Rightarrow \beta$-stable regenerative set R_i (closed range of β-stable subordinator).
- If $\beta > 1/2$, $R_1 \cap R_2$ is a β'-stable R, $\beta' := 2\beta - 1$.



- Kingman (1973): $\forall$ β-stable regenerative set R, $\exists$ measurable mappings $L_t : \mathbf{F} \to \mathbb{R}$, s.t. $L_t(R) \stackrel{fdd}{=}$ inverse β-stable subordinator, $t \geq 0$.
(**F**: space of closed sets on $[0, \infty)$ equipped with Fell topology.)
  - Call L_t *local time functionals*.
- With $a(N) \sim N^{\beta'}$, L_t : local time functionals for β'-stable R,

$$\frac{1}{a(N)} |R_1(N) \cap R_2(N) \cap [0, t]| \Rightarrow L_t(R_1 \cap R_2)$$

  - In actual proof, need joint convergence to $(L_t(R_i \cap R_j),\ i \neq j)$.

Main result

Recall $X(n) = \int'_{E\times E} f(T^n x_1, T^n x_2) M(dx_1)M(dx_2)$, $\beta > 1/2$, $\beta' = 2\beta - 1$.

- B., Owada & Wang (2018): as $N \to \infty$, we have

$$\frac{1}{c(N)} \sum_{1\le n\le Nt} X(n) \Rightarrow$$

$$(\mu\times\mu)(f) \int_{\Omega'\times[0,\infty)} L_t\left(\left(R(\omega_1') + v_1\right) \cap \left(R(\omega_2') + v_2\right) \right) S_\alpha(d\omega_1', dv_1) S_\alpha(d\omega_2', dv_2)$$

in $D[0,1]$ under the uniform topology, where $c(N) \sim N^{\beta'(1-1/\alpha)+1/\alpha}$.

(The limit process above admits a continuous version.)

- $(\Omega', \mathcal{F}', P')$: a probability space;
- $S_\alpha(d\omega', dv)$: SαS random measure with control measure $P' \times v^{-\beta}dv$;
- L_t: local time functionals for β'-stable regenerative set;
- R : a β-stable regenerative set (starting at origin) under P';

  $\bar{R} := \bar{R} + v$ under $P' \times v^{-\beta}dv$ is stationary, i.e., $(\bar{R} - t) \cap [0,\infty) \stackrel{d}{=} \bar{R}$ (Fitzsimmons & Taksar, 1988).

Precise assumption on flow $\{T^n\}$

- Transfer (or dual) operator: $\widehat{T} : L^1(\mu) \to L^1(\mu)$, characterized by the relation:

$$\int_E (\widehat{T} f)(x) \cdot g(x)\mu(dx) = \int_E f(x) \cdot (g \circ T)(x)\mu(dx), \ \forall g \in L^\infty(\mu).$$

Interpretation: f is density of random element $X \Rightarrow \widehat{T} f$ is density of $T(X)$.

- Under ergodicity,

$$\{T^n\} \text{ is conservative } \iff \sum_{n=1}^{\infty} \widehat{T}^n f = \infty \ a.e. \ \forall f \geq 0, \ \mu(f) > 0. \qquad (*)$$

- Assumption: $\exists b_n \sim n^{1-\beta}$, for a rich class of f with $\mathrm{supp}(f) \subset A$,

$$\lim_{n\to\infty} b_n(\widehat{T}^n f)(x) = \mu(f) = \int_A f(x)\mu(dx) \quad \text{uniformly for a.e. } x \in A.$$

This implies $\sum_{n=1}^{N} \widehat{T} f \sim N^\beta$ a.e. on A, i.e., providing divergence rate in (*).

- Comparison: weaker conditions in Owada & Samorodnitsky (2015) involve

$$\lim_{n\to\infty} \frac{b_n}{n} \sum_{k=1}^{n} (\widehat{T}^k f) = \mu(f).$$

Example: interval map (Thaler, 2000)

Fix $\beta \in (0,1)$ ($\beta > 1/2$ for our limit theorem).

- $T: [0,1] \to [0,1]: \ T_\beta(x) = x\left(1 + \left(\frac{x}{1+x}\right)^{1/\beta - 1} - x^{1/\beta-1}\right)^{\beta/(\beta-1)} \mod 1;$
- $\mu(dx) = \left(\frac{1}{x^{1/\beta}} + \frac{1}{(1+x)^{1/\beta}}\right) 1_{(0,1]}(x)dx;$
- $A = [\epsilon, 1], \ \epsilon \in (0,1);$
- f : Riemann integrable on A^2.

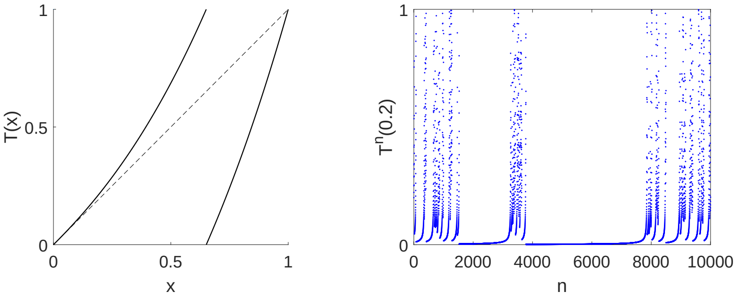


Figure: $\beta = 0.75$

Thank you for your attention!