Excursion landscape

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Notations

- $B = (B_t, t \geq 0)$ is a one-dimensional standard Brownian motion
- $B^{br} = (B_t^{br}, 0 \leq t \leq 1)$ is a Brownian bridge
- $B^{ex} = (B_t^{ex}, 0 \leq t \leq 1)$ is a Brownian excursion
- $B^{me} = (B_t^{me}, 0 \leq t \leq 1)$ is a Brownian meander

That is

$$\begin{aligned} B^{br} &\stackrel{(d)}{=} (B_t, 0 \leq t \leq 1 | B_1 = 0) \\ B^{ex} &\stackrel{(d)}{=} (B_t, 0 \leq t \leq 1 | B_t > 0 \text{ for } 0 < t < 1 \text{ and } B_1 = 0) \\ B^{me} &\stackrel{(d)}{=} (B_t, 0 \leq t \leq 1 | B_t > 0 \text{ for } 0 < t \leq 1) \end{aligned}$$

Bessel Processes and the χ_k^2-distribution

- Let $(R_t^{(k)}, t \geq 0)$ denote a BES(k) process ($k \geq 0$) whose square is the sum of squares of k independent copies of $(B_t, t \geq 0)$ with the usual additive structure for all real $k \geq 0$
- So the one-dimensional distributions of $R^{(k)}$ are determined for all k:

$$R_t^{(k)} \stackrel{(d)}{=} t^{1/2} \chi_k$$

where $\chi_0 = 0$ and χ_k^2 has gamma $(k/2, 1/2)$ distribution, for $k > 0$:

$$\frac{\mathbb{P}(\chi_k \in dx)}{dx} = \frac{x^{k-1} e^{-\frac{1}{2}x^2} \mathbb{1}(x > 0)}{2^{\frac{k}{2}-1} \Gamma(\frac{k}{2})}$$

Depth process

Regard $X[S,T] := (X_t, S \le t \le T)$ as a random landscape
The depth process of X on $[S,T]$ is

$$
\begin{aligned}
D[S,T] &:= (D_t, S \le t \le T) \text{ with} \\
D_t &:= W_t - X_t \ge 0 \quad \text{where} \\
W_t &:= (\max_{S \le s \le t} X_s) \wedge (\max_{t \le s \le T} X_s) \quad \text{for } S \le t \le T
\end{aligned}
$$

is the water level process with drains at S and T.

- Let $\mu := \arg\max X[S,T]$ (the watershed). Note: $D_S = D_\mu = D_T$.
- $W[S,\mu] \uparrow$ as the past max of $X[S,T]$ on $[S,\mu]$.
- $W[\mu,T] \downarrow$ as the future max of $X[S,T]$ on $[\mu,T]$.

The point (t, X_t) in the original landscape is then submerged at depth D_t below the water level $W_t = X_t + D_t$.

Brownian scaling

For S and T with $\mathbb{P}(S < T) = 1$, map $B_t, S \le t \le T$ to

$$B_*[S,T] := (B_*[S,T]_u, 0 \le u \le 1) := \left(\frac{B_{S+u(T-S)}}{\sqrt{T-S}}, 0 \le u \le 1 \right)$$

For $T > 0$ a fixed time (or a random time independent of B), let (g_T, d_T) be the excursion interval of B away from 0 which straddles T.

Examples for basic Brownian landscapes X over $[0,1]$:

1. $B_*[0, g_T] \stackrel{(d)}{=} B^{br}$
2. $|B_*[g_T, d_T]| \stackrel{(d)}{=} B^{ex} \stackrel{(d)}{=}$ BES(3) bridge of length 1 from 0 to 0
3. $|B_*[g_T, T]| \stackrel{(d)}{=} B^{me}$

Bertoin-Pitman (1994) surveys path transformations.

Depth process with drains (Definition)

Given a collection of n distinct drain points $U_1, \ldots, U_n$ with $0 < U_i < 1$, let

$$0 = U_{n,0} < U_{n,1} < \cdots < U_{n,n} < U_{n,n+1} = 1$$

be $\{0, U_1, ..., U_n, 1\}$ in increasing order.

The depth process of X on $[0, 1]$ with drains at $U_1, ..., U_n$ is the process D_n on $[0, 1]$ which on each interval $[U_{n,i}, U_{n,i+1}]$ equals the depth process of X on $[U_{n,i}, U_{n,i+1}]$. In particular, we let

$D_0(X)$ be the depth process of $X := X[0, 1]$ with drains at 0 and 1.
$D_n(X)$ be the depth process with additional n drains at $U_i, 1 \leq i \leq n$ (in addition to drains at 0 and 1).

$B^{br} \leftrightarrow B^{ex}$ (Vervaat)

- According to Vervaat's transformation, a cyclic shift of increments of B^{ex} to start them at time U_1 gives a copy of B^{br} whose minimum value is $-B^{ex}(U_1)$. So $B^{ex}(U_1) \stackrel{(d)}{=} -\min_{0 \le t \le 1} B^{br} \stackrel{(d)}{=} \max_{0 \le t \le 1} B^{br}$
- The Vervaat shift involves a repetitive landscape defined by periodic extension of $-B^{ex}$ to the reals.
- If this repetitive landscape is viewed from the point $(U, -B^{ex}(U))$ and independent of B^{ex}, this landscape becomes a periodic extension of a copy of B^{br} from which $1 - U$ may be recovered as $\arg\max B^{br}$.
- Recall $B^{ex} \stackrel{(d)}{=} R_3^{br}$, where R_3^{br} denotes the law of the BES(3) bridge.

Corollary

$$(\arg\max B^{br}, \max B^{br}) \overset{(d)}{=} (U, B^{ex}(U)) \overset{(d)}{=} (U, R_3^{br}(U)) \overset{(d)}{=} (U, \frac{1}{2}\chi_2)$$

Conditioning on $\arg\max B^{br} = t$,

$$\max B^{br} \overset{(d)}{=} R_3^{br}(t) \overset{(d)}{=} B_3^{br}(t) \overset{(d)}{=} \sqrt{t(1-t)}\chi(3),$$

where B_3^{br} is a 3-dimensional Brownian bridge. Therefore,

$$(\arg\max B^{br}, \max B^{br}) \overset{(d)}{=} (U, \sqrt{U(1-U)}\chi(3)),$$

where U and $\chi(3)$ are independent.

A path transformation

Theorem (Pitman-Yor, 2003)

For a standard Brownian motion B*, let* $\overline{B}_t : \max_{0 \le u \le t} B_u$*, and let* $R_t := \overline{B}_t - B_t$*. Let* (g_s, d_s) *be the excursion interval of* R *away from 0 that straddles time* s*. Let* $M_s := \max_{g_s \le u \le d_s} R_u$ *be the maximum of* R *over this excursion interval, and*

$$V_t := \int_0^t \mathbb{1}(M_s \le \overline{B}_s)ds \quad \text{and} \quad \alpha_u := \inf\{t : V_t > u\}.$$

Then the process $(B_{\alpha_u}, u \ge 0)$ *is a BES(3).*

Informal proof

- BES(3) process can be reconstructed from its excursions below its running maximum
- BES(3) can be viewed as Brownian motion conditioned to stay positive
- The excision operation just converts the PPP of Brownian excursions into the "right" PPP for BES(3) excursions.
- The point process of excursions of BES(3) below its past maximum process is identical in law to the point process obtained from the excursions of B below its past maximum process by deletion of every excursion whose depth exceeds its starting level, meaning that the path of B hits zero during that excursion interval.

Excision of excursions

- Now, we excise all excursions below the waterline which reaches 0. We consider a Brownian motion stopped at T_x, conditioned by $T_x = t$, where T_x is the first hitting time of x.
- The excursion is kept if and only if the depth is $< y$, the excursion is excised if and only if its depth is $\geq y$.

$$
\begin{aligned}
& \mathbb{P}[\text{excised excursion} | y, \zeta] \\
= \; & \mathbb{P}[\max_{[0,\zeta]} \{\text{BES}(3) \text{ bridge with lifetime } \zeta\} \geq y] \\
= \; & \mathbb{P}[\max_{[0,1]} \{\text{BES}(3) \text{ bridge with lifetime } 1\} \geq \frac{y}{\sqrt{\zeta}}] =: F(y/\sqrt{\zeta})
\end{aligned}
$$

- Taking level y and lifetime ζ, we get two independent PPP with respective intensities $\frac{dy d\zeta}{\sqrt{2\pi}\zeta^{3/2}}(F(y/\sqrt{\zeta}))$ for excised excursion, and $\frac{dy d\zeta}{\sqrt{2\pi}\zeta^{3/2}}(1 - F(y/\sqrt{\zeta}))$ for non-excised excursion.

The total lifetime Z_x of the excised excursions

The law of the excised Brownian bridge stopped at T_x, conditioning on $\arg\max B^{br} = t$ and $\max B^{br} = x$ can be computed as follows:

$$
\begin{aligned}
& \mathbb{E}[G((B_s)_{s\le T_x} \text{ excised})| T_x = t] \\
= \quad & \mathbb{E}[G((R_s^{(3)})_{s\le T_x})| T_x + Z_x = t] \\
= \quad & \frac{\mathbb{E}[G((R_s^{(3)})_{s\le T_x}), Z_x = t - T_x]}{\mathbb{P}[Z_x + T_x = t]} \\
= \quad & \frac{\mathbb{E}[G((R_s^{(3)})_{s\le T_x}) D_{Z_x}(t - T_x)]}{\mathbb{E}[D_{Z_x}(t - T_x)]},
\end{aligned}
$$

where D_{Z_x} is the density of the law of the total lifetime Z_x of excised excursions.

This conditional law has density proportional to $D_{Z_x}(t - T_x)$ with respect to the law of a BES(3) process up to T_x.

The law of Z_x is the sum of ζ for a PPP with intensity $\frac{dy d\zeta}{\sqrt{2\pi}\zeta^{3/2}}(F(y/\sqrt{\zeta}))$.