Fractal Properties of Operator Semistable Lévy Processes

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Based on joint works with D. Khoshnevisan,P. Kern, T. Luks, M. M. Meerschaert

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Outline

1. Fractal geometry of stochastic processes: an overview
2. Lévy processes, operator semistable laws
3. Asymptotic bounds for semistable Lévy exponents
4. Hausdorff and packing dimensions of the image and graph
5. Hausdorff dimension of the multiple points

1. Fractal geometry of stochastic processes: an overview

Give a stochastic process $X = \{X(t),\ t \in \mathbb{R}\}$ with values in $\mathbb{R}^d$, it is interesting to study various geometric properties of the following random sets:

- Range $\quad X([0,1]) = \{X(t) :\ t \in [0,1]\}$
- Graph $\quad \mathrm{Gr}X([0,1]) = \{(t, X(t)) :\ t \in [0,1]\}$
- Level set $\quad X^{-1}(x) = \{t \in \mathbb{R}_+ : X(t) = x\}$
- Excursion set $X^{-1}(F) = \{t \in \mathbb{R}_+ : X(t) \in F\}$, $\forall F \subset \mathbb{R}^d$,
- Set of k-multiple points $M_k = \Big\{x \in \mathbb{R}^d$ such that

$$x = X(t_1) = \cdots = X(t_k) \text{ for } t_1 < \cdots < t_k\Big\}.$$

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- If the sample function $t \mapsto X(t)$ is non-differentiable, tools from fractal geometry such as Hausdorff dimension, packing dimension, are useful.
- Some of the problems are naturally related to hitting probabilities and potential theory of the process X.
- Many of the problems have been well studied for Brownian motion, Lévy processes, Gaussian processes, etc.
- New geometric problems continue to arise, not only in the studies of fine & local properties of non-smooth processes, but also in studies of discrete random objects or asymptotic properties as time or space variable goes to infinity. For example, together with Davar Khoshnevisan and Kunwoo Kim, we have been studying macroscopic multifractal analysis of SPDEs.

Two approaches for determining Hausdorff dimension of a random set

Let $F \subset \mathbb{R}^d$. In order to show $\dim_{\rm H} F = c$, there are mainly two approaches:

1. Prove $\dim_{\rm H} F \le c$ by using a covering argument; and $\dim_{\rm H} F \ge c$ by constructing a suitable measure on F.
2. If there is a family of processes $\{Y_\gamma(t), t \ge 0\}$ for which we can characterize their hitting probabilities:

$$\mathbb{P}\Big\{\exists t > 0,\ Y_\gamma(t) \in E\Big\} = 0 \Longleftrightarrow \mathcal{C}_{d-\gamma}(E) = 0, \quad (1)$$

where $\mathcal{C}_\alpha$ denotes the Riesz capacity, then

$$\dim_{\rm H} F = d - \inf\Big\{\gamma > 0:\ Y_\gamma(\mathbb{R}_+) \cap F \neq \emptyset\Big\}.$$

Results of the form (1) have been established for

- Lévy processes (in particular, stable Lévy processes),
- Multiparameter Markov processes: Evans (1987), Fitzsimmons and Salisbury (1989), Hirsch and Song (1995).
- Additive Lévy processes: Khoshnevisan and X. (2002, 2003, 2009).

The rest of talk is built on the work in Khoshnevisan and X. (2002, 2005, 2009), Khoshnevisan, Zhong and X. (2003).

2. Lévy processes, operator semistable laws

Let $\{X(t), t \geq 0\}$ be a Lévy process with values in $\mathbb{R}^d$. For $t > 0$, the characteristic function of $X(t)$ is given by

$$\mathbb{E}\left[e^{i\langle \xi, X(t)\rangle}\right] = e^{-t\Psi(\xi)},$$

where Ψ is the characteristic or Lévy exponent of X, given by

$$\begin{aligned}\Psi(\xi) =& i\langle \mathbf{a}, \xi\rangle + \frac{1}{2}\langle \xi, \Sigma \xi^{'}\rangle \\ & + \int_{\mathbb{R}^d}\left[1 - e^{i\langle x, \xi\rangle} + \frac{i\langle x, \xi\rangle}{1 + \|x\|^2}\right] \nu(dx),\end{aligned}$$

where $\mathbf{a} \in \mathbb{R}^d$ is fixed, Σ is a non-negative definite, symmetric, $d \times d$ matrix, and ν is a Borel measure on $\mathbb{R}^d \setminus \{0\}$ that satisfies

$$\int_{\mathbb{R}^d} \min\{1, \|x\|^2\}\, \nu(dx) < \infty.$$

Operator semistable processes

The Lévy process X is called *operator semistable* if the distribution μ of $X(1)$ is *full*, i.e. not supported on any lower dimensional hyperplane, and fulfills

$$\mu^c = c^E \mu * \varepsilon_u \tag{2}$$

for some fixed $c > 1$, $u \in \mathbb{R}^d$ and some linear operator E on $\mathbb{R}^d$, where μ^c denotes the c-fold convolution power of μ, $c^E\mu(dx) = \mu(c^{-E}dx)$ is the image measure of μ under the exponential operator $c^E = \sum_{n=0}^{\infty} \frac{(\log c)^n}{n!} E^n$, and ε_u denotes the Dirac measure at the point $u \in \mathbb{R}^d$.

Operator semistable distributions were introduced by Jajte (1977), extending the earlier work of Sharp (1969) on operator stable laws.

Some remarks

- In case $u = 0$ the distribution μ and the Lévy process X are called *strictly* operator semistable.
- Any *exponent* E is invertible, and any eigenvalue λ of E further fulfills $\Re(\lambda) \geq \frac{1}{2}$.
- A strictly operator semistable Lévy process X is *strictly operator semi-selfsimilar*: For some $c > 1$,

$$\{X(ct),\, t \geq 0\} \overset{\text{fd}}{=} \{c^E X(t),\, t \geq 0\}, \tag{3}$$

  where $\overset{\text{fd}}{=}$ denotes equality of all finite-dimensional distributions.
- If (2) holds for all $c > 0$ (and $u = 0$), then X is called (strictly) operator-stable.

3. Asymptotic bounds for semistable Lévy exponents

Little has been known about the transition density function of a semistable Lévy process $X = \{X(t), t \geq 0\}$. Hence we will apply the second approach to study the Hausdorff dimensions of random sets generated by X, which are determined by the asymptotic behavior of $\Psi(\xi)$ as $\|\xi\| \to \infty$.

Factor the minimal polynomial of E into $f_1(x) \cdots f_p(x)$ such that every root of $f_j(x)$ has real part a_j, where $a_1 < \cdots < a_p$ are the distinct real parts of the eigenvalues of E and $a_1 \geq \frac{1}{2}$ [cf. Meerschaert and Scheffler (2001).

We can decompose $\mathbb{R}^d$ as

$$\mathbb{R}^d = V_1 \oplus \ldots \oplus V_p,$$

where $V_j = \mathrm{Ker}(f_j(E))$ are E-invariant subspaces.

In an appropriate basis, E can be represented as a block-diagonal matrix blue$E = E_1 \oplus \ldots \oplus E_p$, where $E_j : V_j \to V_j$ and every eigenvalue of E_j has real part a_j. Every V_j is an E_j-invariant subspace of dimension $d_j = \dim_{\mathrm{H}} V_j$.

We can write $x \in \mathbb{R}^d$ as $x = x_1 + \cdots + x_p$ and $t^E x = t^{E_1} x_1 + \cdots + t^{E_p} x_p$ with respect to this direct sum decomposition, where $x_j \in V_j$ and $t > 0$.

Asymptotic bounds for $\Psi(\xi)$

Let $a_1 < \cdots < a_p$ be the distinct real parts of the eigenvalues of E, with $a_1 \geq 1/2$, and define $\alpha_i = a_i^{-1}$ so that $2 \geq \alpha_1 > \cdots > \alpha_p$.

Theorem 3.1 [Kern, Meerschaert and X. (2017)]

For every $\varepsilon > 0$ there exists $\tau > 1$ such that for some $K_i = K_i(\varepsilon, \tau)$ and $\|\xi\| > \tau$ we have

$$K_2 \|\xi\|^{-\varepsilon/2} \sum_{i=1}^{p} \|\xi_i\|^{\alpha_i} \leq \Re(\Psi(\xi)) \leq K_1 \|\xi\|^{\varepsilon/4} \sum_{i=1}^{p} \|\xi_i\|^{\alpha_i},$$

$$|\mathrm{Im}(\Psi(\xi))| \leq K_3 \|\xi\|^{\varepsilon/4} \sum_{i=1}^{p} \|\xi_i\|^{\alpha_i}.$$

4. Hausdorff and packing dimensions of the image and graph

Theorem 4.1 [Khoshnevisan, X. and Zhong (2003)]

For an arbitrary Lévy process X in $\mathbb{R}^d$,

$$\dim_{\mathrm{H}} X([0,1]) = \sup\left\{a < d : \int_{\{\|\xi\|\geq 1\}} \Re\left(\frac{1}{1+\psi(\xi)}\right) \frac{d\xi}{\|\xi\|^{d-a}} < \infty\right\}.$$

Theorem 4.2 [Khoshnevisan and X. (2008)]

For any Lévy process X, the packing dimension of $X([0,1])$ is

$$\dim_{\mathrm{P}} X([0\,,1]) = \sup\left\{\eta \geq 0 : \ \liminf_{r\to 0^+} \frac{W(r)}{r^\eta} = 0\right\} \qquad (4)$$

almost surely, the function W is defined by

$$W(r) = \int_{\mathbb{R}^d} \Re\left(\frac{1}{1+\psi(\frac{x}{r})}\right) \frac{1}{\prod_{j=1}^d (1+x_j^2)}\, dx.$$

Combining Theorem 3.1 with Theorems 4.1 and 4.2, we can show

Theorem 4.3 [Kern, Meerschaert and X. (2017)]

Let X be a strictly operator semistable Lévy process in $\mathbb{R}^d$. Then almost surely

$$\dim_{H} X([0,1]) = \dim_{P} X([0,1]) = \begin{cases} \alpha_1 & \text{if } \alpha_1 \le d_1, \\ 1 + \alpha_2(1 - \alpha_1^{-1}) & \text{else.} \end{cases} \quad (5)$$

Similarly, the Hausdorff and packing dimensions of the graph $\mathrm{Gr}X([0,1])$ can be determined.

5. Hausdorff dimension of the multiple points

A point $x \in \mathbb{R}^d$ is called a *k-multiple point* of the stochastic process X for some integer $k \geq 2$ if there exist $0 \leq t_1 < \cdots < t_k$ such that

$$X(t_1) = \cdots = X(t_k) = x.$$

Existence of k-multiple points of a general Lévy process X has been investigated by Taylor (1967), Hawkes (1978), LeGall, Rosen and Shieh (1989), who prove sufficient conditions; and by Fitzsimmons and Salisbury (1989), Khoshnevisan and Xiao (2002, 2009), who prove necessary and sufficient conditions.

Denote by $M_X(k)$ the set of all k-multiple points of X. This set is a random fractal and its Hausdorff and packing dimensions have been determined when X is a Brownian motion or a strictly stable process in $\mathbb{R}^d$.

Theorem 5.1 [Luks and X. (2016)]

Let $X = \{X(t),\, t \ge 0\}$ be a symmetric, absolutely continuous Lévy process. Then $\dim_{\rm H} M_X(k) = d - \gamma$ almost surely, where

$$\gamma = \inf\left\{\beta \in (0,d] : \int_{\mathbb{R}^{kd}} \frac{d\bar{\xi}}{\left\|\sum_{j=1}^{k} \xi^{(j)}\right\|^{\beta}} \prod_{j=1}^{k} \frac{1}{1+\Psi(\xi^{(j)})} < \infty\right\}.$$

Multiple points

Applying Theorem 5.1, Luks and X. (2016) determined the Hausdorff dimension of $M_X(2)$ of a symmetric operator stable process in $\mathbb{R}^d$.

With the help of Theorem 3.1, Kern, Meerschaert and X. (2017) extended their result to the case of symmetric operator semistable processes.

To state their result, we rearrange the distinct real parts $\alpha_1 > \cdots > \alpha_p$ of the eigenvalues of the exponent E as blue$\tilde{\alpha}_1 \geq \cdots \geq \tilde{\alpha}_d$ including their multiplicities.

Theorem 5.2 [Kern, Meerschaert and and X. (2017)]

Let $X = \{X(t),\ t \geq 0\}$ be a symmetric operator semistable Lévy process in $\mathbb{R}^d$ with exponent E.

(a) If $d = 2$ then

$$\dim_{\mathrm{H}} M_X(2) = \min\left\{\tilde{\alpha}_1\Big(2 - \frac{1}{\tilde{\alpha}_1} - \frac{1}{\tilde{\alpha}_2}\Big),\ 2\tilde{\alpha}_2\Big(1 - \frac{1}{\tilde{\alpha}_1}\Big)\right\}.$$

(b) If $d = 3$ then

$$\dim_{\mathrm{H}} M_X(2) = \tilde{\alpha}_1\left(2 - \frac{1}{\tilde{\alpha}_1} - \frac{1}{\tilde{\alpha}_2} - \frac{1}{\tilde{\alpha}_3}\right).$$

(c) If $d \geq 4$ then $M_X(2) = \emptyset$.

Recently, Luks and X. (2017+) determined the Hausdorff dimension of $M_X(3)$ when it is nonempty.

Thank you !