Potential theory of subordinate killed Brownian motions

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AMS meeting, Indiana University, April 2, 2017

References

This talk is based on the following paper with Panki Kim and Zoran Vondracek:

Potential theory of subordinate killed Brownian motions. Preprint, 2016.

Outline

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Let $W = (W_t, \mathbb{P}_x)$ be a Brownian motion in $\mathbb{R}^d$, $d \geq 1$, and let $S = (S_t)_{t\geq 0}$ be an independent subordinator with Laplace exponent ϕ. The process $X = (X_t, \mathbb{P}_x)$ defined by $X_t = W_{S_t}$, $t \geq 0$, is called a subordinate Brownian motion. It is an isotropic Lévy process with characteristic exponent $\Psi(\xi) = \phi(|\xi|^2)$ and generator $-\phi(-\Delta)$.

In recent years, isotropic, and more generally, symmetric, Lévy processes have been intensively studied and many important results have been obtained. In particular, under certain weak scaling conditions on the characteristic exponent Ψ (or the Laplace exponent ϕ), it was shown that non-negative harmonic functions with respect to these Lévy processes satisfy the scale invariant Harnack inequality (HI) and the scale invariant boundary Harnack principle (BHP).

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If D is an open subset of $\mathbb{R}^d$, we can kill the process X upon exiting D and obtain a process X^D known as a killed subordinate Brownian motion. Functions that are harmonic in an open subset of D with respect to X^D are defined only on D, but by extending them to be identically zero on $\mathbb{R}^d \setminus D$, the HI and BHP follow directly from those for X. In particular, the BHP for X^D is in fact a special case of the BHP for X in D.

By reversing the order of subordination and killing, one obtains a process different from X^D. Assume from now on that D is a domain (i.e., connected open set) in $\mathbb{R}^d$, and let $W^D = (W_t^D, \mathbb{P}_x)$ be the Brownian motion W killed upon exiting D. The process $Y^D = (Y_t^D, \mathbb{P}_x)$ defined by $Y_t^D = W_{S_t}^D$, $t \geq 0$, is called a subordinate killed Brownian motion. It is a symmetric Hunt process with infinitesimal generator $-\phi(-\Delta_{|D})$, where $\Delta_{|D}$ is the Dirichlet Laplacian.

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This process is very natural and useful. For example, it was used in [Chen-Song, JFA 2005] as a tool to obtain two-sided estimates of the eigenvalues of the generator of X^D.

Despite its usefulness, the potential theory of subordinate killed Brownian motions has been studied only sporadically, see [Glover-Rao-Sikic-Song, 94], [Song-Vondracek, PTRF 03], [Glover-PopStojanovic-Rao-Sikic-Song-Vondracek, JFA 04] for stable subordinators, and [Song-Vondracek, JTP 06], and [Song-Vondracek, LNM 09] for more general subordinators.

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In particular, [Song-Vondracek, LNM 09] contains versions of HI and BHP (with respect to the subordinate killed Brownian motion in a bounded Lipschitz domain D) which are very weak in the sense that the results are proved only for non-negative functions which are harmonic in all of D. Those results are easy consequences of the fact that there is a one-to-one correspondence between non-negative harmonic functions (in all of D) with respect to W^D and those with respect to Y^D.

Additionally, some aspects of potential theory of subordinate killed Brownian motions in unbounded domains were recently studied in [Kim-Song-Vondracek, SPA 16].

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In the PDE literature, the operator $-(-\Delta|_D)^{\alpha/2}$, $\alpha \in (0,2)$, which is the generator of the subordinate killed Brownian motion via an $\alpha/2$-stable subordinator, also goes under the name of spectral fractional Laplacian. This operator has been of interest to quite a few people in the PDE circle.

For instance, a version of HI was also shown in [Stinga-Zhang, DCDS 13].

The results in our recent paper are proved for general subordinator satisfying certain condition. For simplicity, in this talk I will only present the results for stable subordinators, that is, for the case $\phi(\lambda) = \lambda^{\alpha/2}$, $\alpha \in (0,2]$. Even in this case, our results are new.

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Outline

For the Harnack inequality, we need some assumptions on the domain D. We will say that a decreasing function $f : (0,\infty) \to (0,\infty)$ satisfies the doubling property if, for every $T > 0$, there exists a constant $c > 0$ such that $f(t) \leq cf(2t)$ for all $t \in (0, T]$.

(B1) The function $t \mapsto \mathbb{P}_x(t < \tau_D^W)$ satisfies the doubling property (with a doubling constant independent of $x \in D$).

(B2) There exist constants $c \geq 1$ and $M \geq 1$ such that for all $t \leq 1$ and $x, y \in D$,

$$c^{-1}\,\mathbb{P}_x(t < \tau_D^W)\,\mathbb{P}_y(t < \tau_D^W)\,t^{-d/2}e^{-\frac{M|x-y|^2}{t}}$$
$$\leq p^D(t,x,y) \leq c\,\mathbb{P}_x(t < \tau_D^W)\,\mathbb{P}_y(t < \tau_D^W)\,t^{-d/2}e^{-\frac{|x-y|^2}{Mt}}.$$

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If D is either a bounded Lipschitz domain or an unbounded domain consisting of all the points above the graph of a globally Lipschitz function, then **(B1)** and **(B2)** are satisfied, cf. (0.36) and (0.25) of [Varopoulos, CJM, 2003].

It is also easy to show that a $C^{1,1}$ domain with compact complement also satisfies **(B1)** and **(B2)**.

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It is also easy to show that a $C^{1,1}$ domain with compact complement also satisfies **(B1)** and **(B2)**.

For any Borel set $U \subset D$, let $\tau_U = \tau_U^{Y^D} = \inf\{t > 0 : \ Y_t^D \notin U\}$ be the exit time of Y^D from U.

A non-negative function f defined on D is said to be *harmonic* in an open set $V \subset D$ with respect to Y^D if for every open set $U \subset \overline{U} \subset V$,

$$f(x) = \mathbb{E}_x \left[f(Y^D_{\tau_U}) \right] \qquad \text{for all } x \in U.$$

A non-negative function f defined on D is said to be *regular harmonic* in an open set $V \subset D$ if

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Here is our first main result.

Harnack inequality

Suppose that $D \subset \mathbb{R}^d$ is a domain satisfying **(B1)–(B2)**. There exists a constant $C > 0$ such that for any $r \in (0, 1]$ and $B(x_0, r) \subset D$ and any function f which is non-negative in D and harmonic in $B(x_0, r)$ with respect to Y^D, we have

$$f(x) \le Cf(y), \qquad \text{for all } x, y \in B(x_0, r/2).$$

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Let $D \subset \mathbb{R}^d$ be an open set and let $Q \in \partial D$. We say that D is $C^{1,1}$ near Q if there exist a localization radius $R > 0$, a $C^{1,1}$-function $\varphi = \varphi_Q : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\varphi(0) = 0$, $\nabla\varphi(0) = (0, \dots, 0)$, $\|\nabla\varphi\|_\infty \le \Lambda$, $|\nabla\varphi(z) - \nabla\varphi(w)| \le \Lambda|z - w|$, and an orthonormal coordinate system CS_Q with its origin at Q such that

$$B(Q,R) \cap D = \{y = (\widetilde{y}, y_d) \in B(0,R) \text{ in } CS_Q : \ y_d > \varphi(\widetilde{y})\},$$

where $\widetilde{y} := (y_1, \dots, y_{d-1})$. The pair (R, Λ) will be called the $C^{1,1}$ characteristics of D at Q.

An open set $D \subset \mathbb{R}^d$ is said to be a (uniform) $C^{1,1}$ open set with characteristics (R, Λ) if it is $C^{1,1}$ with characteristics (R, Λ) near every boundary point $Q \in \partial D$.

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Here is our second main result

Boundary Harnack Principle

Suppose that D is a bounded $C^{1,1}$ domain, or a $C^{1,1}$ domain with compact complement or a domain consisting of all the points above the graph of a bounded globally $C^{1,1}$ function. Let (R, Λ) be the $C^{1,1}$ characteristics of D. There exists a constant $C = C(d, \Lambda, R, \alpha) > 0$ such that for any $r \in (0, R]$, $Q \in \partial D$, and any non-negative function f in D which is harmonic in $D \cap B(Q, r)$ with respect to Y^D and vanishes continuously on $\partial D \cap B(Q, r)$, we have

$$\frac{f(x)}{\delta_D(x)} \le C \frac{f(y)}{\delta_D(y)} \qquad \text{for all } x, y \in D \cap B(Q, r/2).$$

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Our third main result is as follows

Boundary Harnack Principle

There exists a constant $b = b(\alpha, d) > 2$ such that, for every open set $E \subset D$ and every $Q \in \partial E \cap D$ such that E is $C^{1,1}$ near Q with characteristics $(\delta_D(Q) \wedge 1, \Lambda)$, the following holds: There exists a constant $C = C(\delta_D(Q) \wedge 1, \Lambda, \alpha, d) > 0$ such that for every $r \le (\delta_D(Q) \wedge 1)/(b+2)$ and every non-negative function f on D which is regular harmonic in $E \cap B(Q, r)$ with respect to Y^D and vanishes on $E^c \cap B(Q, r)$, we have

$$\frac{f(x)}{\delta_E(x)^{\alpha/2}} \le C \frac{f(y)}{\delta_E(y)^{\alpha/2}}, \qquad x, y \in E \cap B(Q, 2^{-6}\kappa^4 r),$$

where $\kappa := (1 + (1+\Lambda)^2)^{-1/2}$.

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Outline

We will use G^{Y^D} to denote the Green function of Y^D:

$$G^{Y^D}(x,y) = c(\alpha)\int_0^\infty p^D(t,x,y)t^{\alpha/2-1}\,dt\,,$$

where $p^D(t,x,y)$ is the transition density of the killed Brownian motion in D.

We will use J^{Y^D} to denote the jumping intensity of Y^D:

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We will use G^{X^D} to denote the Green function of X^D. Sharp two-sided estimates on G^D were first obtained by Chen-Song and Kulczycki in the late 1990's. For example when D is a bounded $C^{1,1}$ open set, it holds that

$$C^{-1}\left(\frac{\delta_D^{\alpha/2}(x)}{|x-y|}\wedge 1\right)\left(\frac{\delta_D^{\alpha/2}(y)}{|x-y|}\wedge 1\right)\frac{1}{|x-y|^{d-\alpha}}$$
$$\leq G^{X^D}(x,y)\leq C\left(\frac{\delta_D^{\alpha/2}(x)}{|x-y|}\wedge 1\right)\left(\frac{\delta_D^{\alpha/2}(y)}{|x-y|}\wedge 1\right)\frac{1}{|x-y|^{d-\alpha}},$$

for some $C>1$.

We will use $J^X(x,y)=j^X(|x-y|)$ to denote the Lévy density of X. Recall that

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Proposition 1

For every $M > 0$, there exists a constant $C = C(M) \geq 1$ such that for all $x, y \in D$ with $|x - y| \leq M$,

$$C^{-1}\left(\frac{\delta_D(x)}{|x-y|} \wedge 1\right)\left(\frac{\delta_D(y)}{|x-y|} \wedge 1\right)\frac{1}{|x-y|^{d-\alpha}}$$

$$\leq G^{Y^D}(x,y) \leq C\left(\frac{\delta_D(x)}{|x-y|} \wedge 1\right)\left(\frac{\delta_D(y)}{|x-y|} \wedge 1\right)\frac{1}{|x-y|^{d-\alpha}}.$$

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Proposition 2

For every $M > 0$, there exists a constant $C = C(M, R, \Lambda) \geq 1$ such that such that for all $x, y \in D$ with $|x - y| \leq M$,

$$C^{-1}\left(\frac{\delta_D(x)}{|x-y|} \wedge 1\right)\left(\frac{\delta_D(y)}{|x-y|} \wedge 1\right)\frac{1}{|x-y|^{d+\alpha}}$$
$$\leq J^{Y^D}(x,y) \leq C\left(\frac{\delta_D(x)}{|x-y|} \wedge 1\right)\left(\frac{\delta_D(y)}{|x-y|} \wedge 1\right)\frac{1}{|x-y|^{d+\alpha}}.$$

Propositions 1 and 2 above imply global two-sided estimates on G^{Y^D} and J^{Y^D} for bounded D, but only give "local" two-sided estimates for unbounded D. The following two results give sharp two-sided estimates for G^{Y^D} and J^{Y^D} when D is an unbounded $C^{1,1}$ domain of the two types above.

Proposition 2

For every $M > 0$, there exists a constant $C = C(M, R, \Lambda) \geq 1$ such that such that for all $x, y \in D$ with $|x - y| \leq M$,

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Theorem 3

(1) Let $D \subset \mathbb{R}^d$ be a domain consisting of all the points above the graph of a bounded globally $C^{1,1}$ function. There exists a constant $C_1 = C_1(R, \Lambda) \geq 1$ such that for all $x, y \in D$,

$$C_1^{-1}\left(\frac{\delta_D(x)}{|x-y|}\wedge 1\right)\left(\frac{\delta_D(y)}{|x-y|}\wedge 1\right)\frac{1}{|x-y|^{d-\alpha}} \leq G^{Y^D}(x,y)$$
$$\leq C_1\left(\frac{\delta_D(x)}{|x-y|}\wedge 1\right)\left(\frac{\delta_D(y)}{|x-y|}\wedge 1\right)\frac{1}{|x-y|^{d-\alpha}}.$$

(2) Let $D \subset \mathbb{R}^d$ be a $C^{1,1}$ domain with compact complement. There exists a constant $C_2 = C_2(R, \Lambda) \geq 1$ such that for all $x, y \in D$,

$$C_2^{-1}\left(\frac{\delta_D(x)}{|x-y|\wedge 1}\wedge 1\right)\left(\frac{\delta_D(y)}{|x-y|\wedge 1}\wedge 1\right)\frac{1}{|x-y|^{d-\alpha}} \leq G^{Y^D}(x,y)$$
$$\leq C_2\left(\frac{\delta_D(x)}{|x-y|\wedge 1}\wedge 1\right)\left(\frac{\delta_D(y)}{|x-y|\wedge 1}\wedge 1\right)\frac{1}{|x-y|^{d-\alpha}}.$$

Theorem 4

(1) Let $D \subset \mathbb{R}^d$ be a domain consisting of all the points above the graph of a bounded globally $C^{1,1}$ function. There exists a constant $C_1 = C_1(R, \Lambda) \geq 1$ such that for all $x, y \in D$,

$$C_1^{-1}\left(\frac{\delta_D(x)}{|x-y|} \wedge 1\right)\left(\frac{\delta_D(y)}{|x-y|} \wedge 1\right)\frac{1}{|x-y|^{d+\alpha}} \leq J^{Y^D}(x,y)$$
$$\leq C_1\left(\frac{\delta_D(x)}{|x-y|} \wedge 1\right)\left(\frac{\delta_D(y)}{|x-y|} \wedge 1\right)\frac{1}{|x-y|^{d+\alpha}}.$$

(2) Let $D \subset \mathbb{R}^d$ be a $C^{1,1}$ domain with compact complement. There exists a constant $C_2 = C_2(R, \Lambda) \geq 1$ such that for all $x, y \in D$,

$$C_2^{-1}\left(\frac{\delta_D(x)}{|x-y| \wedge 1} \wedge 1\right)\left(\frac{\delta_D(y)}{|x-y| \wedge 1} \wedge 1\right)\frac{1}{|x-y|^{d+\alpha}} \leq J^{Y^D}(x,y)$$
$$\leq C_2\left(\frac{\delta_D(x)}{|x-y| \wedge 1} \wedge 1\right)\left(\frac{\delta_D(y)}{|x-y| \wedge 1} \wedge 1\right)\frac{1}{|x-y|^{d+\alpha}}.$$

Choose a $C^{1,1}$-function $\varphi : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\varphi(\widetilde{0}) = 0$, $\nabla\varphi(\widetilde{0}) = (0, \dots, 0)$, $\|\nabla\varphi\|_\infty \le \Lambda$, $|\nabla\varphi(\widetilde{y}) - \nabla\varphi(\widetilde{w})| \le \Lambda|\widetilde{y} - \widetilde{w}|$, and an orthonormal coordinate system CS_Q with its origin at $Q \in \partial D$ such that

$$B(z, R) \cap D = \{y = (\widetilde{y}, y_d) \in B(0, R) \text{ in } CS_Q : y_d > \varphi(\widetilde{y})\}.$$

Define $\rho_Q(x) := x_d - \varphi(\widetilde{x})$, where $(\widetilde{x}, x_d)$ are the coordinates of x in CS_Q. We define for $r_1, r_2 > 0$,

$$D_Q(r_1, r_2) := \left\{y \in D : r_1 > \rho_Q(y) > 0, |\widetilde{y}| < r_2\right\}.$$

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Theorem 5(Carleson Estimate)

There exists a constant $C = C(R, \Lambda) > 0$ such that for every $Q \in \partial D$, $0 < r < R/2$, and every non-negative function f in D that is harmonic in $D \cap B(Q, r)$ with respect to Y^D and vanishes continuously on $\partial D \cap B(Q, r)$, we have

$$f(x) \leq Cf(x_0) \qquad \text{for } x \in D \cap B(Q, r/2),$$

where $x_0 \in D \cap B(Q, r)$ with $\rho_Q(x_0) = r/2$.

It is not difficult to see that that there exists $L = L(R, \Lambda, d) > 0$ such that for every $Q \in \partial D$ and $r \leq \kappa R$, one can find a $C^{1,1}$ domain $V_Q(r)$ with characteristics $(rR/L, \Lambda L/r)$ such that $D_Q(r/2, r/2) \subset V_Q(r) \subset D_Q(r, r)$.

Lemma 6

There exists $C = C(R, \Lambda) > 0$ such that for every $r \leq \kappa^{-1} R/2$, $Q \in \partial D$ and $x \in D_Q(r/4, r/4)$,

$$\mathbb{P}_x \left(Y^D(\tau_{V_Q(r)}) \in D_Q(2r, 2r) \right) \leq C \, \frac{\delta_D(x)}{r}.$$

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To prove the second boundary Harnac principle, we will show that when U is relatively compact subset of D, the process $Y^{D,U}$ can be thought of as a non-local Feynman-Kac transform of X^U. Moreover, if U is a certain $C^{1,1}$ domain, the conditional gauge function related to this transform is bounded between two positive constants which will imply that the Green functions of X^U and $Y^{D,U}$ are comparable. We will prove a uniform version of this result in the sense that the comparability constants are independent of the set U as long as its diameter is small and not larger than a multiple of its distance to the boundary.

Let $(\mathcal{E}^{X^U}, \mathcal{D}(\mathcal{E}^{X^U}))$ be the Dirichlet form of X^U. Then,

$$\mathcal{E}^{X^U}(f,f) = c\int_0^\infty \int_U f(x)(f(x) - P_s f(x))\,dx\, s^{-\alpha/2-1} ds\,,$$

$$\mathcal{D}(\mathcal{E}^{X^U}) = \{f \in L^2(U,dx) : \mathcal{E}^{X^U}(f,f) < \infty\}\,,$$

where (P_s) is the Brownian semigroup.

Furthermore, for $f \in \mathcal{D}(\mathcal{E}^{X^U})$,

$$\mathcal{E}^{X^U}(f,f) = \frac{1}{2}\int_U\int_U (f(x)-f(y))^2 J^X(x,y)dydx + \int_U f(x)^2 \kappa_U^X(x)dx,$$

where

$$\kappa_U^X(x) = \int_{\mathbb{R}^d\setminus U} J^X(x,y)dy\,.$$

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The Dirichlet form $(\mathcal{E}^{Y^D}, \mathcal{D}(\mathcal{E}^{Y^D}))$ of Y^D is given by

$$\mathcal{E}^{Y^D}(f,f) = c\int_0^\infty \int_D f(x)(f(x) - P_s^D f(x))\,dx\,s^{-\alpha/2-1}ds$$

and $\mathcal{D}(\mathcal{E}^{Y^D}) = \{f \in L^2(D,dx) : \mathcal{E}^{Y^D}(f,f) < \infty\}$, where (P_t^D) is the semigroup of the killed Brownian motion W^D.

Moreover, for $f \in \mathcal{D}(\mathcal{E}^{Y^D})$,

$$\mathcal{E}^{Y^D}(f,f) = \frac{1}{2}\int_D\int_D (f(x)-f(y))^2 J^{Y^D}(x,y)dydx + \int_D f(x)^2 \kappa^{Y^D}(x)dx,$$

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Hence, the Dirichlet form $(\mathcal{E}^{Y^{D,U}}, \mathcal{D}(\mathcal{E}^{Y^{D,U}}))$ of $Y^{D,U}$ is equal to

$$\mathcal{E}^{Y^{D,U}}(f,f) = s\int_0^\infty \int_U f(x)(f(x) - P_s^D f(x))\, dx\, s^{-\alpha/2-1} ds\,,$$

$$\mathcal{D}(\mathcal{E}^{Y^{D,U}}) = \{f \in L^2(U,dx) :\ \mathcal{E}^{Y^{D,U}}(f,f) < \infty\}\,.$$

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where

$$\kappa_U^{Y^D}(x) = \kappa^{Y^D}(x) + \int_{D\setminus U} J^{Y^D}(x,y)dy\,.$$

Lemma 7

For $x, y \in D$,

$$J^X(x,y) - J^{Y^D}(x,y) \le j^X(\delta_D(y)).$$

Lemma 8

Let U be a relatively compact open subset of D. Then $\mathcal{D}(\mathcal{E}^{X^U}) = \mathcal{D}(\mathcal{E}^{Y^{D,U}})$.

For $x, y \in D$, $x \neq y$, let

$$F(x,y) := \frac{J^{Y^D}(x,y)}{J^X(x,y)} - 1 = \frac{J^{Y^D}(x,y) - J^X(x,y)}{J^X(x,y)},$$

and $F(x,x) = 0$. Then $-1 < F(x,y) \le 0$.

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Lemma 9

There exists $b = b(\phi, d) > 2$ such that for all $x_0 \in D$ and all $r \in (0, 1/b)$ satisfying $B(x_0, (b+1)r) \subset D$, we have that

$$\sup_{x,y \in B(x_0,r)} |F(x,y)| \leq \frac{1}{2} .$$

Let $b > 2$ be the constant from Lemma 9. For $r < 1/b$, let $U \subset D$ be such that $\mathrm{diam}(U) \leq r$ and $\mathrm{dist}(U, \partial D) \geq (b+2)r$. Then there exists a ball $B(x_0, r)$ such that $U \subset B(x_0, r)$ and $B(x_0, (b+1)r) \subset D$. By Lemma 9 we see that

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$$|F(x,y)| \leq 1/2 \qquad \text{for all } x, y \in U.$$

Hence we can define the non-local multiplicative functional

$$K_t^U = \exp\left(\sum_{0<s\le t} \log(1+F(X_{s-}^U, X_s^U))\right).$$

Let

$$T_t^U f(x) := \mathbb{E}_x[K_t^U f(X_t^U)].$$

Then $(T_t^U)_{t\ge 0}$ is a strongly continuous semigroup on $L^2(U,dx)$ with the associated quadratic form $(\mathcal{Q}, \mathcal{D}(\mathcal{E}^{X^U}))$ where

$$\mathcal{Q}(f,f) = \mathcal{E}^{X^U}(f,f) - \int_U\int_U f(x)f(y)F(x,y)J^X(x,y)\,dy\,dx.$$

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Lemma 10

For $r < 1/b$, let $U \subset D$ be such that $\mathrm{diam}(U) \le r$ and $\mathrm{dist}(U, \partial D) \ge (b+2)r$. Then

$$(\mathcal{Q}, \mathcal{D}(\mathcal{E}^{X^U})) = (\mathcal{E}^{Y^{D,U}}, \mathcal{D}(\mathcal{E}^{Y^{D,U}}))\,.$$

For $x, y \in U$, $x \neq y$, let

$$u^U(x,y) := \mathbb{E}_x^y[K^U_{\tau_U^X}]$$

be the conditional gauge function for K^U_t. Since $F \le 0$, we have $\log(1+F) \le 0$, hence $K^U_{\tau_U^X} \le 1$. Therefore, $u^U(x,y) \le 1$. Define

$$V^U(x,y) = u^U(x,y) G_U^X(x,y), \quad x, y \in U.$$

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$$V^U(x,y) = u^U(x,y) G^X_U(x,y), \quad x, y \in U.$$

It follows from the theory of conditional gauges that $V^U(x, y)$ is the Green function for the semigroup $(T_t^U)_{t\geq 0}$. Combining this with Lemma 10 we can conclude that V^U is equal to the Green function $G_U^{Y^D}$ of $Y^{D,U}$. Therefore,

$$G_U^{Y^D}(x, y) = u^U(x, y) G_U^X(x, y), \quad x, y \in U.$$

We could also show that for $C^{1,1}$ open sets U, the conditional gauge function u^U is bounded below by a strictly positive constant uniform in the diameter of U. Together with the above equality this proves that the Green function of $Y^{D,U}$ is comparable to that of X^U.

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Thank you!