Compatibility of change of measures

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- Given several probability measures $Q_1, \ldots, Q_n$ and distribution measures $F_1, \ldots, F_n$, does there exist a random variable $X : \Omega \to \mathbb{R}$ such that X has distribution F_i under Q_i for $i = 1, \ldots, n$?

- $(\Omega, \mathcal{A})$: measurable space
- $\mathcal{F}$: the set of distributions on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$,
- $\mathcal{P}$: the set of probability measures on $(\Omega, \mathcal{A})$
- $D_{\mathrm{KL}}(\cdot||\cdot)$: Kullback-Leibler divergence between probability measures

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Definition (Compatibility)

$(F_1, \ldots, F_n) \in \mathcal{F}^n$ and $(Q_1, \ldots, Q_n) \in \mathcal{P}^n$ are *compatible* if there exists a random variable X in $(\Omega, \mathcal{A})$ such that for each $i = 1, \ldots, n$, the distribution of X under Q_i is F_i.

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Definition (Almost compatibility)

$(F_1, \ldots, F_n) \in \mathcal{F}^n$ and $(Q_1, \ldots, Q_n) \in \mathcal{P}^n$ are *almost compatible*, if for any $\epsilon > 0$, there exists a random variable X_ϵ in $(\Omega, \mathcal{A})$ such that for each $i = 1, \ldots, n$, the distribution of X_ϵ under Q_i, denoted by $F_{i,\epsilon}$, is absolutely continuous with respect to F_i, and satisfies $D_{\mathrm{KL}}(F_{i,\epsilon}||F_i) < \epsilon$.

Definition (Convex order)

Let $(\Omega_1, \mathcal{A}_1, P_1)$ and $(\Omega_2, \mathcal{A}_2, P_2)$ be two probability spaces. For $\mathbf{X} \in L_1^n(\Omega_1, \mathcal{A}_1, P_1)$ and $\mathbf{Y} \in L_1^n(\Omega_2, \mathcal{A}_2, P_2)$, we write $\mathbf{X}|_{P_1} \prec_{\text{cx}} \mathbf{Y}|_{P_2}$, if $\mathbb{E}^{P_1}[f(\mathbf{X})] \leq \mathbb{E}^{P_2}[f(\mathbf{Y})]$ for all convex functions $f : \mathbb{R}^n \to \mathbb{R}$, provided that both expectations exist.

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Lemma ()

For $\mathbf{X} \in L_1^n(\Omega_1, \mathcal{A}_1, P_1)$ *and* $\mathbf{Y} \in L_1^n(\Omega_2, \mathcal{A}_2, P_2)$, $\mathbf{X}|_{P_1} \prec_{\text{cx}} \mathbf{Y}|_{P_2}$ *if and only if there exist a probability space* $(\Omega_3, \mathcal{A}_3, P_3)$ *and* $\mathbf{X}', \mathbf{Y}' \in L_1^n(\Omega_3, \mathcal{A}_3, P_3)$ *such that* $\mathbf{X}'|_{P_3} \overset{\text{d}}{=} \mathbf{X}|_{P_1}$, $\mathbf{Y}'|_{P_3} \overset{\text{d}}{=} \mathbf{Y}|_{P_2}$, *and* $\mathbb{E}^{P_3}[\mathbf{Y}'|\mathbf{X}'] = \mathbf{X}'$.

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Conclution: $Q_1, \ldots, Q_n$ are more *variabile* than $F_1, \ldots, F_n$

Necessary condition for compatibility

Lemma

Let $(F_1, \ldots, F_n) \in \mathcal{F}^n$ and $(Q_1, \ldots, Q_n) \in \mathcal{P}^n$. If $(F_1, \ldots, F_n)$ and $(Q_1, \ldots, Q_n)$ are compatible, then

(i) *For any $F \in \mathcal{F}$, $F_i \ll F$ for $i = 1, ..., n$, there exists $Q \in \mathcal{P}$, $Q_i \ll Q$ for $i = 1, ..., n$, such that*

$$\left(\frac{\mathrm{d}F_1}{\mathrm{d}F}, \ldots, \frac{\mathrm{d}F_n}{\mathrm{d}F}\right)\Bigg|_F \prec_{\mathrm{cx}} \left(\frac{\mathrm{d}Q_1}{\mathrm{d}Q}, \ldots, \frac{\mathrm{d}Q_n}{\mathrm{d}Q}\right)\Bigg|_Q . \tag{1}$$

(ii) *For any $Q \in \mathcal{P}$, $Q_i \ll Q$ for $i = 1, ..., n$, there exists $F \in \mathcal{F}$, $F_i \ll F$ for $i = 1, ..., n$, such that (1) holds.*

- $(\Omega, \mathcal{A}) = ([0,1], \mathcal{B}([0,1]))$
- Q_1: probability point mass at 0
- Q_2: probability point mass at 1
- F_1 and F_2: uniform distribution on $[0,1]$

There exists $Q = \frac{1}{2}(Q_1 + Q_2)$ and $F = F_1$ such that

$$\left.\left(\frac{\mathrm{d}F_1}{\mathrm{d}F}, \frac{\mathrm{d}F_2}{\mathrm{d}F}\right)\right|_F \prec_{\mathrm{cx}} \left.\left(\frac{\mathrm{d}Q_1}{\mathrm{d}Q}, \frac{\mathrm{d}Q_2}{\mathrm{d}Q}\right)\right|_Q,$$

but (Q_1, Q_2) and (F_1, F_2) are not compatible.

Theorem

Suppose that $(Q_1, \ldots, Q_n) \in \mathcal{P}^n$, $(F_1, \ldots, F_n) \in \mathcal{F}^n$ *and* $(\Omega, \mathcal{A}, Q_i)$ *is atomless for each* $i = 1, \ldots, n$. $(Q_1, \ldots, Q_n)$ *and* $(F_1, \ldots, F_n)$ *are almost compatible if and only if there exist* $F \in \mathcal{F}$ *and* $Q \in \mathcal{P}$, *such that* $F_i \ll F$, $Q_i \ll Q$ *for* $i = 1, \ldots, n$, *and*

$$\left(\frac{\mathrm{d}F_1}{\mathrm{d}F}, \ldots, \frac{\mathrm{d}F_n}{\mathrm{d}F}\right)\Bigg|_F \prec_{\mathrm{cx}} \left(\frac{\mathrm{d}Q_1}{\mathrm{d}Q}, \ldots, \frac{\mathrm{d}Q_n}{\mathrm{d}Q}\right)\Bigg|_Q. \tag{2}$$

- $(\Omega, \mathcal{A})$: $([0,1], \mathcal{B}(\mathbb{R}))$
- $Q_2 = \lambda$, $\frac{dQ_1}{dQ_2}(t) = 2t$, $t \in [0,1]$
- $F_2 = \lambda$ on $[0,1]$, $\frac{dF_1}{dF_2}(x) = |4x-2|$, $x \in [0,1]$

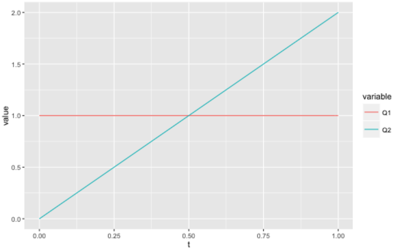
$\Rightarrow$

- $\frac{dQ_1}{dQ_2}$ uniform on $[0,2]$ under Q_2
- $\frac{dF_1}{dF_2}$ uniform on $[0,2]$ under F_2

Taking $Q = Q_2$ and $F = F_2$,

$$\left(\frac{dF_1}{dF}, \frac{dF_2}{dF}\right)\bigg|_F \overset{d}{=} \left(\frac{dQ_1}{dQ}, \frac{dQ_2}{dQ}\right)\bigg|_Q .$$

However, (Q_1, Q_2) and (F_1, F_2) are not compatible.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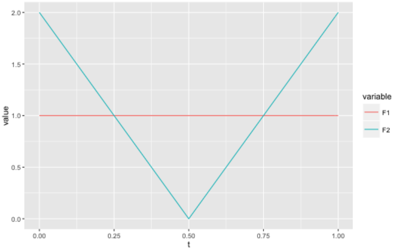
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Theorem

Let $(F_1, \ldots, F_n) \in \mathcal{F}^n$ and $(Q_1, \ldots, Q_n) \in \mathcal{P}^n$. If $(F_1, \ldots, F_n)$ and $(Q_1, \ldots, Q_n)$ are compatible, then there exist $F \in \mathcal{F}$ and $Q \in \mathcal{P}$ such that $F_i \ll F$, $Q_i \ll Q$ for $i = 1, \ldots, n$, and

$$\left(\frac{\mathrm{d}F_1}{\mathrm{d}F}, \ldots, \frac{\mathrm{d}F_n}{\mathrm{d}F}\right)\Bigg|_F \prec_{\mathrm{cx}} \left(\frac{\mathrm{d}Q_1}{\mathrm{d}Q}, \ldots, \frac{\mathrm{d}Q_n}{\mathrm{d}Q}\right)\Bigg|_Q . \tag{3}$$

Conversely, assume there exist $F \in \mathcal{F}$ and $Q \in \mathcal{P}$ such that $F_i \ll F$, $Q_i \ll Q$ for $i = 1, \ldots, n$, and (3) holds. If in addition, there exists a continuous random variable defined on $(\Omega, \mathcal{A}, Q)$, independent of $\left(\frac{\mathrm{d}Q_1}{\mathrm{d}Q}, \ldots, \frac{\mathrm{d}Q_n}{\mathrm{d}Q}\right)$, then $(F_1, \ldots, F_n)$ and $(Q_1, \ldots, Q_n)$ are compatible.

- I: a closed interval
- $C(I)$: the space of all continuous functions defined on I
- $\mathcal{C}_I$: the cylindrical σ-field
- $\mathcal{G}_I$: the set of probability measures on $(C(I), \mathcal{C}_I)$

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Definition

For a closed interval $I \subset \mathbb{R}$, $(G_1, \dots, G_n) \in \mathcal{G}_I^n$ and $(Q_1, \dots, Q_n) \in \mathcal{P}^n$ are *compatible* if there exists a continuous stochastic process $X = \{X(t)\}_{t \in I}$ defined on $(\Omega, \mathcal{A})$ such that for each $i = 1, \dots, n$, the distribution of X under Q_i is G_i.

Theorem

Assume there exist $G \in \mathcal{G}_I$ *and* $Q \in \mathcal{P}$ *such that* $G_i \ll G$, $Q_i \ll Q$ *for* $i = 1, \ldots, n$, *and*

$$\left(\frac{\mathrm{d}G_1}{\mathrm{d}G}, \ldots, \frac{\mathrm{d}G_n}{\mathrm{d}G}\right)\Bigg|_F \prec_{\mathrm{cx}} \left(\frac{\mathrm{d}Q_1}{\mathrm{d}Q}, \ldots, \frac{\mathrm{d}Q_n}{\mathrm{d}Q}\right)\Bigg|_Q$$

holds. If in addition, there exists a continuous random variable defined on $(\Omega, \mathcal{A}, Q)$ *independent of* $\left(\frac{\mathrm{d}Q_1}{\mathrm{d}Q}, \ldots, \frac{\mathrm{d}Q_n}{\mathrm{d}Q}\right)$, *then* $(G_1, \ldots, G_n)$ *and* $(Q_1, \ldots, Q_n)$ *are compatible.*

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- P: a probability measure
- $B = \{B_t\}_{t \in [0,T]}$: P-standard Brownian motion

The Girsanov Theorem says that, by defining Q_θ via

$$\frac{\mathrm{d}Q_\theta}{\mathrm{d}P} = e^{\theta B_T - \frac{\theta^2}{2} T}, \tag{4}$$

$\tilde{B}(t) = B(t) - \theta t$ is a Brownian motion under Q_θ.

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- (G_0, G_μ) and (P, Q_θ) are compatible ?

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- (G_0, G_μ) and (P, Q_θ) are compatible ?

Proposition

Let $P \in \mathcal{P}$ and $B = \{B_t\}_{t \in [0,T]}$ be a P-standard Brownian motion. Using the above notation, for $\mu, \theta \in \mathbb{R}$, (P, Q_θ) and (G_0, G_μ) are compatible if and only if $|\mu| \leq |\theta|$.