Functional limit laws for recurrent excited random walks

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Joint work with Elena Kosygina

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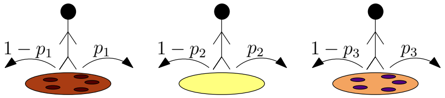
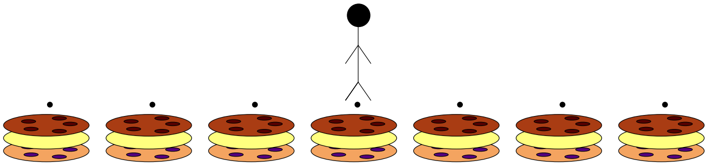
Excited (Cookie) Random Walks

Cookie Environment

- *M* cookies at each site.
- Cookie *strengths* $p_1, p_2, \ldots, p_M \in (0, 1)$.
- Eating a cookie induces a drift for the next step.





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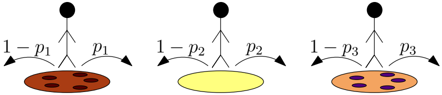

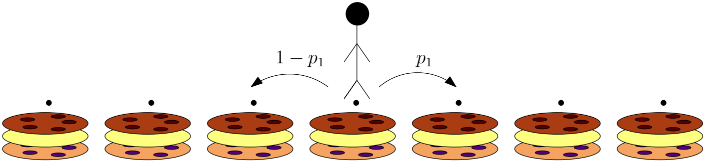

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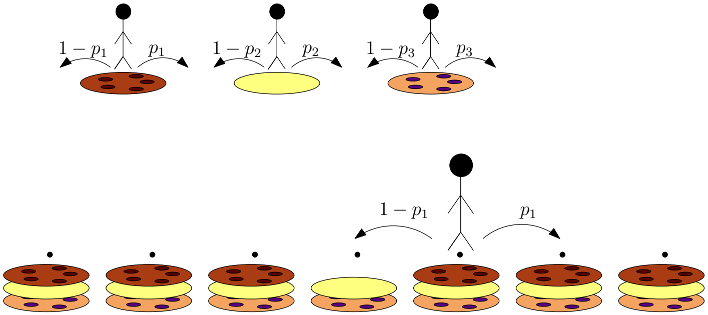

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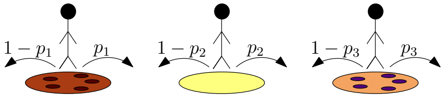

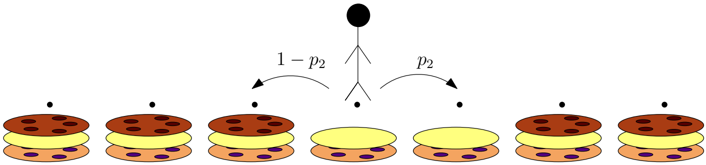

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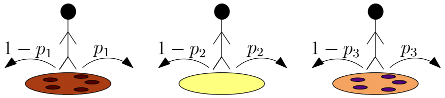

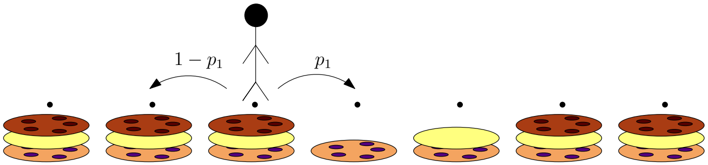

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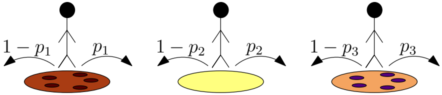

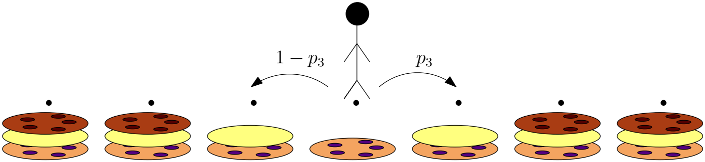

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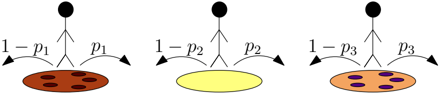

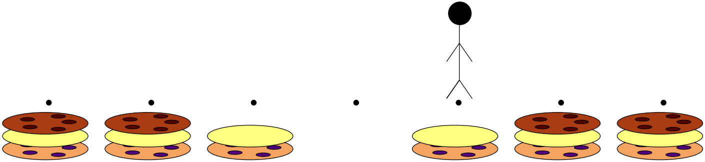

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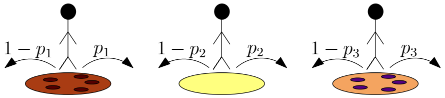

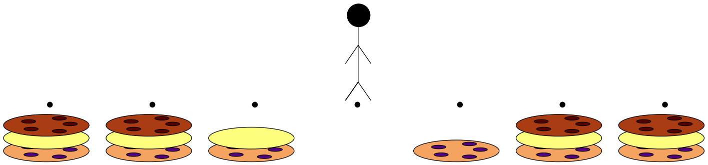

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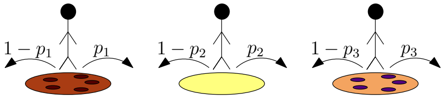

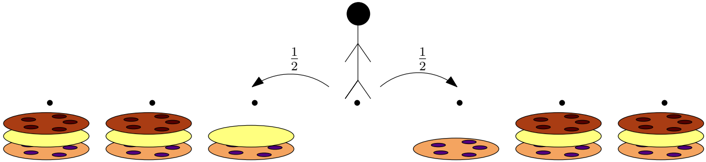

Recurrence/Transience

Total drift per site $$\delta = \sum_{j=1}^{M} (2p_j - 1)$$

Theorem (Zerner '05, Zerner & Kosygina '08)

- *If $\delta > 1$ then $X_n \to +\infty$.*
- *If $\delta < -1$ then $X_n \to -\infty$.*
- *If $\delta \in [-1, 1]$ then X_n is recurrent.*

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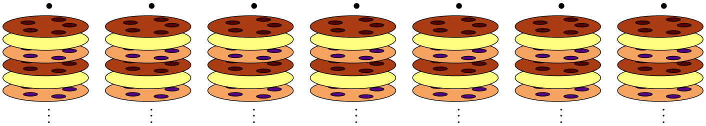
Other results involving δ. (B. S. '08, K. Z. '08, K. M. '11)

- **Limiting speed:** positive speed $\iff \delta > 2$.
- **Limiting distributions:**
  - CLT if $\delta > 4$
  - $\frac{\delta}{2}$-stable limits if $\delta \in (2, 4)$.

Periodic cookie stacks

- Periodic cookie sequence $p_1, p_2, \ldots, p_M, p_1, p_2, \ldots$



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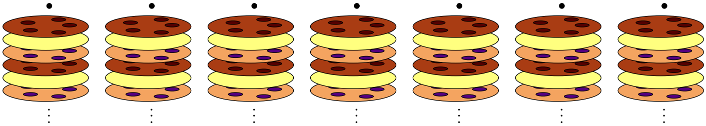

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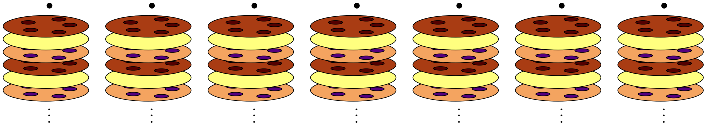

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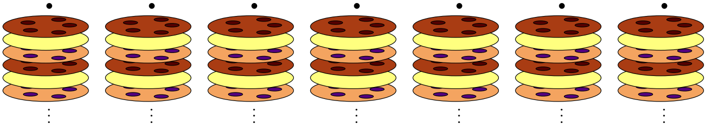

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Note: $\lim_{n\to\infty}\sum_{j=1}^{n}(2p_j - 1)$ doesn't exist.

Periodic cookie stacks - recurrence/transience

$$\theta = \frac{\sum_{j=1}^{M} \sum_{i=1}^{j} (1-p_j)(2p_i - 1)}{2\sum_{j=1}^{M} p_j(1-p_j)} \quad \text{and} \quad \tilde{\theta} = \frac{\sum_{j=1}^{M} \sum_{i=1}^{j} p_j(1-2p_i)}{2\sum_{j=1}^{M} p_j(1-p_j)}.$$

Theorem (Kozma, Shinkar, Orenshtein '16)

For excited random walks with periodic cookie stacks and $\bar{p} = \frac{1}{2}$,

- $\theta > 1$ *implies* $X_n \to +\infty$.
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- $\max\{\theta, \tilde{\theta}\} \leq 1$ *implies* X_n *is recurrent.*

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Other results involving θ and $\tilde{\theta}$. (Kosygina & Peterson '15)

- **Limiting speed:** Non-zero speed if $\theta > 2$ or $\tilde{\theta} > 2$.
- **Transient limiting distributions:** CLT if $\theta > 4$ or $\tilde{\theta} > 4$.

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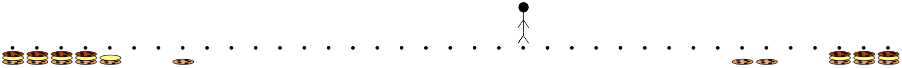

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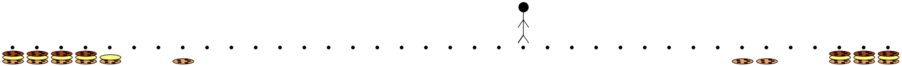

- Like Brownian motion in interior of range
- Gets additional drift at boundary of range

Perturbed Brownian Motion

(α, β)-perturbed Brownian motion

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Theorem (Dolgopyat & Kosygina '12)

For ERW with M cookies per site, if $\delta \in (-1, 1)$ then $\left\{ \frac{X_{nt}}{\sqrt{n}} \right\}_{t \ge 0}$ converges to a $(\delta, -\delta)$-perturbed Brownian motion.

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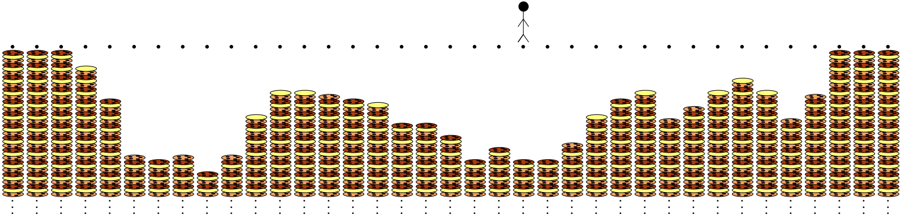

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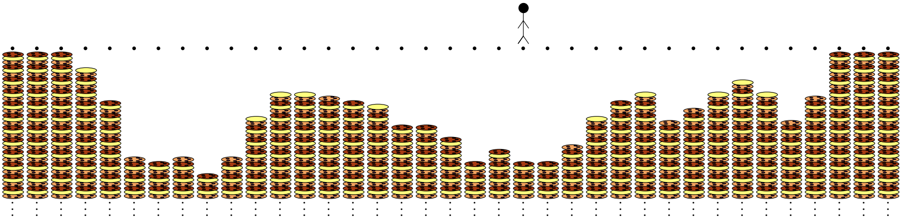

Theorem (Kosygina & Peterson '16)

For ERW with periodic cookie stacks, if $\theta, \tilde{\theta} < 1$ *then* $\left\{ \frac{X_{nt}}{a\sqrt{n}} \right\}_{t \geq 0}$ *converges to a* $(\theta, \tilde{\theta})$*-perturbed Brownian motion, with* $a = \frac{1}{2} \left(\frac{1}{M} \sum_{j=1}^{M} p_j(1 - p_j) \right)^{-1/2} > 1$.

Sketch of proof (M-cookie case)

$$X_n = B_n + C_n, \qquad \text{where} \quad C_n = \sum_{k=0}^{n-1} E[X_{k+1} - X_k \mid \mathcal{F}_k].$$

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Step 3: $\frac{X_{nt}}{\sqrt{n}}$ is tight. Thus, any limit of

$$\frac{X_{nt}}{\sqrt{n}} \approx \frac{B_{nt}}{\sqrt{n}} + \delta \frac{M_{nt}}{\sqrt{n}} - \delta \frac{I_{nt}}{\sqrt{n}}$$

must be a $(\delta, -\delta)$-perturbed Brownian motion.

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Step 2: $C_n \approx \rho(M_n - X_n) + \tilde{\rho}(I_n - X_n)$.

$$X_n \approx B_n + \rho M_n + \tilde{\rho} I_n - (\rho + \tilde{\rho}) X_n$$

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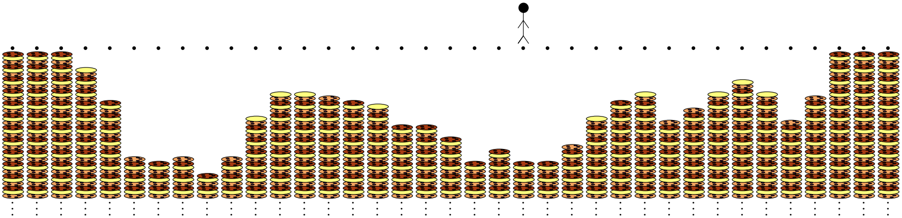
Thus,

$$\begin{aligned} X_n &\approx \frac{B_n}{1+\rho+\tilde{\rho}} + \frac{\rho}{1+\rho+\tilde{\rho}} M_n + \frac{\tilde{\rho}}{1+\rho+\tilde{\rho}} I_n \\ &= \tilde{B}_n + \theta M_n + \tilde{\theta} I_n. \end{aligned}$$

Understanding $C_n \approx \rho(M_n - X_n) + \tilde{\rho}(I_n - X_n)$.



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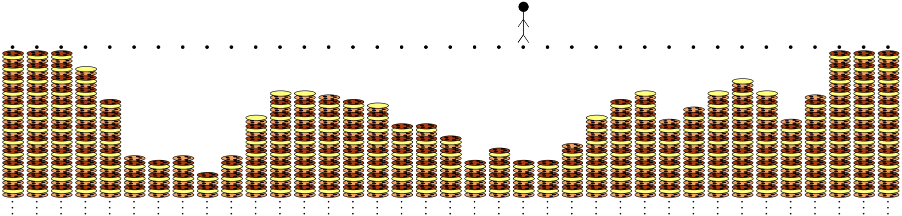

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- If $X_n < x \leq M_n$ then last step from x was to the **left.**
- If $I_n \leq x < X_n$ then last step from x was to the **right.**
- Generate steps from a single site.

$$\rho = \lim_{n\to\infty} E\left[\text{drift consumed until } n\text{-th step } \textbf{left}\right]$$

$$\tilde{\rho} = \lim_{n\to\infty} E\left[\text{drift consumed until } n\text{-th step } \textbf{right}\right]$$