Analysis of space-time fractional stochastic partial differential equations

Erkan Nane

Department of Mathematics and Statistics
Auburn University

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Outline

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- Time Fractional SPDEs
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Fractional time derivative: Two popular approaches!

- Riemann-Liouville fractional derivative of order $0 < \beta < 1$;

$$\mathbb{D}_t^{\beta} g(t) = \frac{1}{\Gamma(1-\beta)} \frac{\partial}{\partial t} \left[\int_0^t g(s) \frac{ds}{(t-s)^{\beta}} \right]$$

  with Laplace transform $s^{\beta}\tilde{g}(s)$, $\tilde{g}(s) = \int_0^{\infty} e^{-st} g(t)dt$ denotes the usual Laplace transform of g.

- Caputo fractional derivative of order $0 < \beta < 1$;

$$D_t^{\beta} g(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{dg(s)}{ds} \frac{ds}{(t-s)^{\beta}} \quad (1)$$

  was invented to properly handle initial values (Caputo 1967). Laplace transform of $D_t^{\beta} g(t)$ is $s^{\beta}\tilde{g}(s) - s^{\beta-1}g(0)$ incorporates the initial value in the same way as the first derivative.

examples

- $$D_t^\beta(t^p) = \frac{\Gamma(1+p)}{\Gamma(p+1-\beta)} t^{p-\beta}$$
- $$D_t^\beta(e^{\lambda t}) = \lambda^\beta e^{\lambda t} - \frac{t^{-\beta}}{\Gamma(1-\beta)}?$$
- $$D_t^\beta(\sin t) = \sin(t + \frac{\pi\beta}{2})$$

Space-Time fractional PDE

The solution to the equation (Nigmatullin 1986; Zaslavsky 1994)

$$\partial_t^\beta u(t,x) = -\nu(-\Delta)^{\alpha/2} u(t,x); \quad u(0,x) = u_0(x), \tag{2}$$

where ∂_t^β is the Caputo fractional derivative of index $\beta \in (0,1)$ and $\alpha \in (0,2]$ is given by

$$\begin{aligned} u(t,x) &= \mathbb{E}_x(u_0(Y(E_t))) = \int_0^\infty P(s,x) f_{E_t}(s) ds \\ &= \int_{\mathbb{R}^d} \left(\int_0^\infty p(s,x-y) f_{E_t}(s) ds \right) u_0(y) dy \end{aligned} \tag{3}$$

where $f_{E_t}(s)$ is the density of inverse stable subordinator of index $\beta \in (0,1)$, and Y is α-stable Lévy process. Here $P(t,x) = \mathbb{E}_x(u_0(Y(t)))$ is the semigroup of α-stable Lévy process.

Time fractional spde model?

We want to study the equations of the following type

$$\partial_t^\beta u(t,x) = -\nu(-\Delta)^{\alpha/2} u(t,x) + \lambda\sigma(u)\,\dot{W}\,(t,x); \quad u(0,x) = u_0(x), \tag{4}$$

where $\dot{W}(t,x)$ is a space-time white noise with $x \in \mathbb{R}^d$.
Assume that $\sigma(\cdot)$ satisfies the following global Lipschitz condition, i.e. there exists a generic positive constant *Lip* such that :

$$|\sigma(x) - \sigma(y)| \leq Lip|x - y| \quad \text{for all} \quad x,\ y \in \mathbb{R}. \tag{5}$$

Assume also that the initial datum is $L^p(\Omega)$ bounded $(p \geq 2)$, that is

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}|u_0(x)|^p < \infty. \tag{6}$$

Time fractional Duhamel's principle

Let $G(t,x)$ be the fundamental solution of the time fractional PDE $\partial^\beta u = L_x u$. The solution to the time-fractional PDE with force term $f(t,x)$

$$\partial_t^\beta u(t,x) = L_x u(t,x) + f(t,x); \quad u(0,x) = u_0(x), \tag{7}$$

is given by Duhamel's principle (Umarov and Saydmatov, 2006), the influence of the external force $f(t,x)$ to the output can be count as

$$\partial_t^\beta V(\tau,t,x) = L_x V(\tau,t,x); \quad V(\tau,\tau,x) = \partial_t^{1-\beta} f(t,x)|_{t=\tau}, \tag{8}$$

which has solution

$$V(t,\tau,x) = \int_{\mathbb{R}^d} G(t-\tau, x-y)\partial_\tau^{1-\beta} f(\tau,x)dx$$

Hence solution to (7) is given by

$$u(t,x) = \int_{\mathbb{R}^d} G(t,x-y)u_0(y)dy + \int_0^t \int_{\mathbb{R}^d} G(t-r,x-y)\partial_r^{1-\beta} f(r,x)dxdr.$$

Model?

Hence if we use this approach we will get the solution of

$$\partial_t^\beta u(t,x) = L_x u(t,x) + \dot{W}(t,x); \quad u(0,x) = u_0(x), \tag{9}$$

to be of the form (informally):

$$\begin{aligned} u(t,x) &= \int_{\mathbb{R}^d} G(t,x-y)u_0(y)dy \\ &\quad + \int_0^t \int_{\mathbb{R}^d} G(t-r,x-y)\partial_r^{1-\beta}[\dot{W}(r,y)]dydr.! \end{aligned} \tag{10}$$

here I am not sure what the fractional derivative in the Walsh-Dalang integral mean?

Another point is that the stochastic integral maybe non-Gaussian!?

Let $\gamma > 0$, define the fractional integral by

$$I_t^\gamma f(t) := \frac{1}{\Gamma(\gamma)} \int_0^t (t-\tau)^{\gamma-1} f(\tau) d\tau.$$

For every $\beta \in (0,1)$, and $g \in L^\infty(\mathbb{R}_+)$ or $g \in C(\mathbb{R}_+)$

$$\partial_t^\beta I_t^\beta g(t) = g(t).$$

Therefore if we consider the time fractional PDE with a force given by $f(t,x) = I_t^{1-\beta} g(t,x)$, then by the Duhamel's principle the solution to (7) will be given by

$$\begin{aligned} u(t,x) &= \int_{\mathbb{R}^d} G(t,x-y) u^0(y) dy \\ &+ \int_0^t \int_{\mathbb{R}^d} G(t-r,x-y) \partial_r^{1-\beta} [I_r^{1-\beta} g(r,x)] dx dr \\ &= \int_{\mathbb{R}^d} G(t,x-y) u^0(y) dy + \int_0^t \int_{\mathbb{R}^d} G(t-r,x-y) g(r,x) dx dr. \end{aligned} \tag{11}$$

"Correct" TFSPDE Model

We should consider the following model problem (Here $L_x = -\nu(-\Delta)^{\alpha/2}$):

$$\partial_t^\beta u(t,x) = L_x u(t,x) + \lambda I_t^{1-\beta}[\sigma(u)\, \dot{W}\,(t,x)];\ \ u(0,x) = u_0(x), \quad (12)$$

By the Duhamel's principle, mentioned above, (12) will have solution

$$\begin{aligned} u(t,x) = &\int_{\mathbb{R}^d} G(t, x-y) u_0(y) dy \\ &+ \lambda \int_0^t \int_{\mathbb{R}^d} G(t-r, x-y)\sigma(u(r,y)) W(dydr). \end{aligned} \quad (13)$$

These space-time fractional SPDEs may arise naturally by considering the heat equation in a material with thermal memory, Chen et al. (2015).

The fractional integral above in equation (12) for functions $\phi \in L^2(\mathbb{R}^d)$ is defined as

$$\int_{\mathbb{R}^d} \phi(x) I_t^{1-\beta}[\dot{W}(t,x)]dx = \frac{1}{\Gamma(1-\beta)} \int_{\mathbb{R}^d} \int_0^t (t-\tau)^{-\beta} \phi(x) W(d\tau dx),$$

is well defined only when $0 < \beta < 1/2$!

An important reason to take the fractional integral of the noise in equation (12): Apply the fractional derivative of order $1-\beta$ to both sides of the equation (12) to see the forcing function, in the traditional units x/t (B. Baeumer et al. (2005); K. Li et al. (2011) and (2013)).

$G(t,x)$ is the density function of $Y(Q_t)$, where Y is an isotropic α-stable Lévy process in $\mathbb{R}^d$ and Q_t is the first passage time of a β-stable subordinator $D = \{D_r,\ r \geq 0\}$. Let $p_{Y(s)}(x)$ and $f_{Q_t}(s)$ be the density of $Y(s)$ and Q_t, respectively. Then the Fourier transform of $p_{Y(s)}(x)$ is given by

$$\widehat{p_{Y(s)}}(\xi) = e^{-s\nu|\xi|^\alpha}, f_{Q_t}(x) = t\beta^{-1}x^{-1-1/\beta}g_\beta(tx^{-1/\beta}), \tag{14}$$

where $g_\beta(\cdot)$ is the density function of D_1. By conditioning, we have

$$G(t,x) = \int_0^\infty p_{Y(s)}(x) f_{Q_t}(s) ds. \tag{15}$$

Lemma 1. Let $d < \min\{2, \beta^{-1}\}\alpha$, then

$$\int_{\mathbb{R}^d} G^2(t,x)dx = t^{-\beta d/\alpha} \frac{(\nu)^{-d/\alpha} 2\pi^{d/2}}{\alpha\Gamma(\frac{d}{2})} \frac{1}{(2\pi)^d} \int_0^\infty z^{d/\alpha-1}(E_\beta(-z))^2 dz \tag{16}$$

Theorem (Mijena and N., 2015)

Let $d < \min\{2, \beta^{-1}\}\alpha$. If σ is Lipschitz continuous and u_0 is measurable and bounded, then there exists a continuous random field $u \in \cup_{\gamma>0}\mathcal{L}^{\gamma,2}$ that solves (12) with initial function u_0. Moreover, u is a.s.-unique among all random fields that satisfy the following: There exists a positive and finite constant L-depending only on Lip, and $\sup_{z\in\mathbb{R}^d}|u_0(z)|$- such that

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}\left(|u_t(x)|^k\right) \leq L^k \exp(Lk^{1+\alpha/(\alpha-\beta d)}t) \qquad (17)$$

In contrast to SPDEs, TFSPDEs have random field (function) solutions for $d < \min\{2, \beta^{-1}\}\alpha$.
We use Picard iteration and a stochastic Young inequality to prove this Theorem.

Curse of Dimensionality for SPDEs

- Consider the SPDE,

$$\partial_t Z(t,x) = \frac{\nu}{2}\Delta Z(t,x) + \dot{W}(t,x) \quad (t > 0,\ x \in \mathbb{R}^d,\ d \geq 2)$$

subject to $Z(0,x) = 0$.

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- The weak solution is

$$Z(t,x) = \int_{(0,t)\times\mathbb{R}} p_{t-s}(y-x)W(dsdy),$$

where $p_t(x) = (2\nu\pi t)^{-d/2}\exp\{-||x||^2/(2\nu t)\}$.

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- Not a random function; If it were then it would be a GRF with

$$E(|Z(t,x)|^2) = \int_0^t ds \int_{\mathbb{R}^d} dy[p_s(y)]^2 = \int_0^t p_{2s}(0)ds \approx \int_0^t s^{-d/2} ds = \infty. \tag{18}$$

Finite energy solution

Random field u is a finite energy solution to the stochastic heat equation (12) when $u \in \cup_{\gamma>0}\mathcal{L}^{\gamma,2}$ and there exists $\rho_* > 0$ such that

$$\int_0^\infty e^{-\rho_* t}\mathbb{E}(|u_t(x)|^2)dt < \infty \quad \text{for all} \quad x \in \mathbb{R}^d.$$

If $\rho \in (0,\infty)$, then

$$\int_0^\infty e^{-\rho t}\mathbb{E}(|u_t(x)|^2)dt \le [\mathcal{N}_{\gamma,2}(u)]^2 \cdot \int_0^\infty e^{-(\rho-2\gamma)t}dt.$$

Therefore if $\rho > 2\gamma$ and $\mathcal{N}_{\gamma,2}(u) = \sup_{t>0,\ x\in\mathbb{R}^d} e^{-\gamma t}\big(\mathbb{E}(|u_t(x)|^2)\big)^{1/2} < \infty$, then the preceding integral is finite. When σ is Lipschitz-continuous function and u_0 is bounded and measurable, then there exists a finite energy solution to the time fractional stochastic type equation (12).

If we drop the assumption of linear growth for σ, then we have the next theorem that extends the result of Foondun and Parshad (2014).

Theorem (Mijena and N., 2015)

Suppose $\inf_{z\in\mathbb{R}^d} u_0(z) > 0$ *and* $\inf_{y\in\mathbb{R}^d} |\sigma(y)|/|y|^{1+\epsilon} > 0$. *Then, there is no finite-energy solution to the time fractional stochastic heat equation* (12).

Hence there is no solution that satisfies

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}(|u_t(x)|^2) \leq L\exp(Lt) \quad \text{for} \quad \text{all} \quad t > 0.$$

Non-existence of global solutions

Fujita (1966), considered

$$\partial_t u_t(x) = \Delta u_t(x) + u_t(x)^{1+\lambda} \quad \text{for} \quad t > 0 \quad x \in \mathbb{R}^d,$$

with initial condition u_0 and $\lambda > 0$. Set $p_c := 2/d$, Fujita showed that for $\lambda < p_c$, the only global solution is the trivial one.
While for $\lambda > p_c$, global solutions exist whenever u_0 is small enough.
At first, this result might seem counterintuitive, but the right intuition is that for large λ, if the initial condition is small, then $u^{1+\lambda}$ is even smaller and the dissipative effect of the Laplacian prevents the solution to grow too big for blow-up to happen.
And when λ is close to zero, then irrespective of the size of the initial condition, the dissipative effect of the Laplacian cannot prevent blow up of the solution.

In our case, we consider $\mathbb{E}|u_t(x)|^2$, there is still an interplay between the dissipative effect of the operator and the forcing term and we are able to shed light only on part of the true picture.
We extend the results of Chow (2009; 2011) to space fractional SPDEs:

Theorem (Foondun, N., Liu (2016))

Suppore that the function σ is a locally Lipschitz function satisfying the following growth condition. There exists a $\gamma > 0$ such that

$$\sigma(x) \geq |x|^{1+\gamma} \quad \text{for all} \quad x \in \mathbb{R}. \tag{19}$$

Suppose also that $\inf_{x\in\mathbb{R}^d} u_0(x) := \kappa$. Let u_t be the solution to stochastic PDE ($\beta = 1$) and suppose that $\kappa > 0$. Then there exists a $t_0 > 0$ such that for all $x \in \mathbb{R}$,

$$\mathbb{E}|u_t(x)|^2 = \infty \quad \text{whenever} \quad t \geq t_0.$$

Intermittency

Definition: The random field $u(t, x)$ is called weakly intermittent if $\inf_{z\in\mathbb{R}^d} |u_0(z)| > 0$, and $\gamma_k(x)/k$ is strictly increasing for $k \geq 2$ for all $x \in \mathbb{R}^d$, where

$$\gamma_k(x) := \liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}(|u(t, x)|^k).$$

Fact: If $\gamma_2(x) > 0$ for all $x \in \mathbb{R}^d$, then $\gamma_k(x)/k$ is strictly increasing for $k \geq 2$ for all $x \in \mathbb{R}^d$.

Theorem (Mijena and N., 2016)

Let $d < \min\{2, \beta^{-1}\}\alpha$. *If* $\inf_{z\in\mathbb{R}^d} |u_0(z)| > 0$, *then*

$$\inf_{x\in\mathbb{R}^d} \gamma_2(x) \geq [C^*(L_\sigma)^2\Gamma(1-\beta d/\alpha)]^{\frac{1}{(1-\beta d/\alpha)}}$$

where $L_\sigma := \inf_{z\in\mathbb{R}^d} |\sigma(z)/z|$. *Therefore, the solution* $u(t,x)$ *of* (12) *is weakly intermittent when* $\inf_{z\in\mathbb{R}^d} |u_0(z)| > 0$ *and* $L_\sigma > 0$.

This theorem extends the results of Foondun and Khoshnevisan (2009) to the time fractional stochastic heat type equations. Recall the constant $C^* = const \cdot \nu^{-d/\alpha}$. Hence Theorem above implies the so-called "very fast dynamo property," $\lim_{\nu\to\infty} \inf_{x\in\mathbb{R}^d} \gamma_2(x) = \infty$. This property has been studied in fluid dynamics: Arponen and Horvai (2007), Baxendale and Rozovskii(1993), Galloway (2003). This theorem is proved by using an application of non-linear renewal theory.

Non-linear noise excitation

Recall our equation
$\partial_t^\beta u(t,x) = L_x u(t,x) + \lambda I_t^{1-\beta}[\sigma(u) \dot{W}(t,x)]; \quad u(0,x) = u_0(x).$
We set

$$\mathcal{E}_t(\lambda) := \sqrt{\int_{\mathbb{R}^d} \mathbb{E}|u_t(x)|^2 \, dx}.$$

and define the nonlinear excitation index by

$$e(t) := \lim_{\lambda\to\infty} \frac{\log\log \mathcal{E}_t(\lambda)}{\log \lambda}$$

This was first studied by Khoshnevisan and Kim (2013). This measures in some sense the effect of non-linear noise on the solution of the SPDE.

The following theorem shows that as the value of λ increases, the solution rapidly develops tall peaks that are distributed over relatively small islands!

Theorem (Foondun and N. 2017)

Fix $t > 0$ and $x \in \mathbb{R}$, we then have

$$\lim_{\lambda \to \infty} \frac{\log \log \mathbb{E}|u_t(x)|^2}{\log \lambda} = \frac{2\alpha}{\alpha - \beta d}.$$

Moreover, if the energy of the solution exists, then the excitation index, $e(t)$ is also equal to $\frac{2\alpha}{\alpha - \beta d}$.

Intermittency fronts

- What if $\inf |u_0(x)| = 0$, say u_0 has compact support.
- We consider a non-random initial function $u_0 : \mathbb{R}^d \to \mathbb{R}$ that is measurable and bounded, has compact support, and is strictly positive on an open subinterval of $(0, \infty)^d$.
- We consider $\alpha = 2$.
- $\sigma : \mathbb{R} \to \mathbb{R}$ Lipschitz and $\sigma(0) = 0$.
- A kind of weak intermittency occur. Roughly, tall peaks arise as $t \to \infty$, but the farthest peaks move roughly linearly with time away from the origin–intermittency fronts.

- Define, for all $p \geq 2$ and for all $\theta \geq 0$,

$$\mathscr{L}_p(\theta) := \limsup_{t\to\infty} \frac{1}{t} \sup_{|x|>\theta t} \log \mathbb{E}\left(|u_t(x)|^p\right). \tag{20}$$

- We can think of $\theta_{L_p} > 0$ as an *intermittency lower front* if $\mathscr{L}_p(\theta) < 0$ for all $\theta > \theta_{L_p}$, and
- of $\theta_{U_p} > 0$ as an *intermittency upper front* if $\mathscr{L}_p(\theta) > 0$ whenever $\theta < \theta_{U_p}$.
- If there exists θ_* that is both a lower front and an upper front then θ_* is the intermittency front–Phase transition.

Theorem (Mijena, N. (2016), Asogwa, N. (2017))

Under the above conditions, the time fractional stochastic heat equation (12) *has a positive intermittency lower front. In fact,*

$$\mathscr{L}_p(\theta) < 0 \quad \text{if } \theta > \frac{p^2}{4}\left(\frac{4\nu}{p}\right)^{1/\beta} (Lip_\sigma c_0)^{2\left(\frac{2-\beta}{2-\beta d}\right)}. \tag{21}$$

In addition, under the cone condition $L_\sigma = \inf_{z\in\mathbb{R}} |\sigma(z)/z| > 0$, *there exists* $\theta_0 > 0$ *such that*

$$\mathscr{L}_p(\theta) > 0 \quad \text{if } \theta \in (0, \theta_0). \tag{22}$$

That is, in this case, the stochastic heat equation has a finite intermittency upper front.

This theorem in the case of the stochastic heat equation (for $p = 2$, and $d = 1$) was proved by Conus and Khoshnevisan (2012).

- We use a version of stochastic Young inequality for the proof of the intermittency lower front.
- Let $d = 1$.
- When $\sigma(x) = Ax$. PAM. Our theorem implies that, if there were an intermittency front, then it would lie between θ_0 and

$$\frac{2^{1/\beta}(Ac_0)^{4/(2-\beta)}}{(Ac_0)^{2\beta/(2-\beta)}}.$$

- When $\beta = 1$. The existence of an intermittency front has been proved recently by Le Chen and Dalang (2012); in fact, they proved that the intermittency front is at $A^2/2$.

TFSPDE in bounded domains with Dirichlet boundary conditions.

How about the moment estimates and growth of the solution of the following TFSPDEs with Dirichlet boundary conditions? (t fixed large λ and λ fixed, large t.)

$$\begin{aligned} \partial_t^\beta u(t,x) &= L_x u(t,x) + \lambda I_t^{1-\beta}[\sigma(u)\, \dot{W}\,(t,x)], \quad x \in B(0,R); \\ u(0,x) &= u^0(x), \end{aligned} \tag{23}$$

Following Walsh (1986) and using the time fractional Duhamel's principle (Umarov 2012), (23) will have (mild/integral) solution

$$\begin{aligned} u(t,x) = & \int_B G_B(t,x-y)u^0(y)dy \\ & + \lambda \int_0^t \int_B G_B(t-r,x-y)\sigma(u(s,y))W(dydr). \end{aligned} \tag{24}$$

Theorem (Foondun, Mijena, N. (2016))

Suppose that $d < (2 \wedge \beta^{-1})\alpha$. Then under Lipschitz condition on σ, there exists a unique random-field solution to (23) satisfying

$$\sup_{x \in B(0,R)} \mathbb{E}|u_t(x)|^2 \leq c_1 e^{c_2 \lambda^{\frac{2\alpha}{\alpha - d\beta}} t} \quad \text{for all} \quad t > 0.$$

Here c_1 and c_2 are positive constants.

Theorem (Foondun, Mijena, N. (2016))

Fix $\epsilon > 0$ and let $x \in B(0, R-\epsilon)$, then for any $t > 0$,

$$\lim_{\lambda\to\infty} \frac{\log\log \mathbb{E}|u_t(x)|^2}{\log \lambda} = \frac{2\alpha}{\alpha - d\beta},$$

where u_t is the mild solution to (23). The excitation index of the solution to (23), $e(t)$ is equal to $\frac{2\alpha}{\alpha-d\beta}$.

This result for $\beta = 1, \alpha \in (1,2)$ is established by Foondun et al (2015), and for $\beta = 1, \alpha = 2$ it is established by Khoshnevisan and Kim (2015) and Foondun and Joseph (2015).

Large time fixed λ behavior of the solution

Recently Foondun and Nualart (2015) studied the long time ($t \to \infty$) behavior of the the second moment of the solution to (23) when $\alpha = 2, \ \beta = 1 \ \ d = 1.$ We have extended their results to $\alpha \in (1, 2)$

Theorem (Foondun, Guerngar, and N. (2017))

There exists $\lambda_0 > 0$ such that for all $\lambda < \lambda_0$ and $x \in B(0, R)$

$$-\infty < \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}|u_t(x)|^2 < 0.$$

On the other hand, for all $\epsilon > 0$, there exists $\lambda_1 > 0$ such that for all $\lambda > \lambda_1$ and $x \in B(0, R - \epsilon)]$,

$$0 < \liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}|u_t(x)|^2 < \infty.$$

Theorem says that the solution grows exponentially when λ is large.

When λ is small the solution decays exponentially.
We define the energy of the solution u as

$$\mathcal{E}_t(\lambda) := \sqrt{\mathbb{E}||u_t||^2}.$$

We have also the following

$$-\infty < \limsup_{t\to\infty} \frac{1}{t} \log \mathcal{E}_t(\lambda) < 0 \quad \text{for all } \lambda < \lambda_0.$$

$$0 < \liminf_{t\to\infty} \frac{1}{t} \log \mathcal{E}_t(\lambda) < \infty \quad \text{for all } \lambda > \lambda_1.$$

Intuitively these results show that if the amount of noise is small, then the heat lost in the system modeled by the Dirichlet equation is not enough to increase the energy in the long run.
On the other hand, if the amount of noise is large enough, then the energy will increase

SPDEs with space-colored noise

Look at the equation with colored noise.

$$\partial_t^\beta u(t,x) = L_x u(t,x) + \lambda I_t^{1-\beta}[\sigma(u)\ \dot{F}\ (t,x)]; \quad u(0,x) = u^0(x), \quad (25)$$

in $(d+1)$ dimensions, where $\nu > 0$, $\beta \in (0,1)$, $\alpha \in (0,2]$, $-(-\Delta)^{\alpha/2}$ is the generator of an isotropic stable process, $\dot{F}\ (t,x)$ is white noise in time and colored in space, and $\sigma : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous, and satisfies $L_\sigma |x| \leq \sigma(x) \leq Lip_\sigma |x|$ with L_σ and Lip_σ being positive constants.

$\dot{F}$ denotes the Gaussian colored noise satisfying the following property, $\mathbb{E}[\dot{F}(t,x)\dot{F}(s,y)] = \delta_0(t-s)f(x,y)$. This can be interpreted more formally as

$$Cov\left(\int \phi dF, \int \psi dF\right) = \int_0^\infty ds \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy \phi_s(x)\psi_s(y)f(x-y) \quad (26)$$

where we use the notation $\int \phi dF$ to denote the wiener integral of ϕ

We will assume that the spatial correlation of the noise term is given by the following function for $\gamma < d$,

$$f(x,y) := \frac{1}{|x-y|^{\gamma}}.$$

Following Walsh (1986), we define the mild solution of (25) as the predictable solution to the following integral equation

$$u_t(x) = (\mathcal{G}u_0)_t(x) + \lambda \int_{\mathbb{R}^d} \int_0^t G_{t-s}(x,y)\sigma(u_s(y))F(dsdy). \tag{27}$$

where

$$(\mathcal{G}u_0)_t(x) := \int_{\mathbb{R}^d} G_t(x,y)u_0(y)dy,$$

and $G_t(x,y)$ is the space-time fractional heat kernel.

Future work/Open problems

1. Large space behavior of STF-SPDEs
2. Employing the STF-SPDEs for modelling data!
3. Inverse problems for SPDEs
4. Existence-non-existence of solutions, blow up of solutions in finite time.
5. Comparison of solutions for different initial values and/or σs!
6. Approximations of SPDEs: what happens when you have time fractional derivatives? For regular diffusions and stable process, one can have random walk approximations and come up with a system of SDEs. Can one follow the same strategy here? We can approximate time fractional diffusions by CTRWs.!

Thank You!

Some References

- S. Asogwa and E. Nane. Intermittency fronts for space-time fractional stochastic partial differential equations in (d+1) dimensions. Stochastic Process. Appl. 127 (2017), no. 4, 1354–1374.
- Zhen-Qing Chen, Kyeong-Hun Kim, Panki Kim. Fractional time stochastic partial differential equations. Stochastic Process Appl. 125 (2015), 1470–1499.
- L. Chen, G. Hu, Y. Hu and J. Huang. Space-time fractional diffusions in Gaussian noisy environment. Stochastics, 2016.
- L. Chen, Y. Hu and D. Nualart. Nonlinear stochastic time-fractional slow and fast diffusion equations on $\mathbb{R}^d$. Preprint, 2016.
- M. Foodun, N. Guerngar and E. Nane. Some properties of non-linear fractional stochastic heat equations on bounded domains. Chaos, Solitons & Fractals (To appear 2017).

- M. Foondun and D. Khoshnevisan, Intermittence and nonlinear parabolic stochastic partial differential equations, Electron. J. Probab. 14 (2009), no. 21.
- M. Foondun, W. Liu, M. Omaba, Moment bounds for a class of fractional stochastic heat equations. 2014
- M. Foodun, J.B. Mijena and E. Nane. Non-linear noise excitation for some space-time fractional stochastic equations in bounded domains. Fract. Calc. Appl. Anal. Vol. 19, No 6 (2016), pp. 1527–1553.
- M Foondun and E. Nane. Moment bounds for a class of space-time fractional stochastic equations. Mathematische Zeitschrift (2017), pp. 1–27.
- M. Foondun,and E. Nualart. On the behaviour of stochastic heat equations on bounded domains. ALEA Lat. Am. J. Probab. Math. Stat. 12 (2015), no. 2, 551–571.

- J. B. Mijena, and E.Nane. Intermittence and time fractional partial differential equations. Potential Anal. Vol. 44 (2016) 295–312.
- J.B. Mijena and E. Nane. Space-time fractional stochastic partial differential equations. Stochastic Process. Appl. 125, (2015) 3301–3326
- Umarov, Sabir; Saydamatov, Erkin A fractional analog of the Duhamel principle. Fract. Calc. Appl. Anal. 9 (2006), no. 1, 57–70.
- S. Umarov, and E. Saydamatov. A fractional analog of the Duhamel principle. Fract. Calc. Appl. Anal. 9 (2006), no. 1, 57–70.
- J.B. Walsh, An introduction to stochastic partial differential equations. Ecole d?Eté de Probabilités de Saint-Flour XIV - 1984, Lect. Notes Math. 1180 (1986) 265–437.

Physical explanation/motivation of the model!

The following discussion is adapted from Chen et al.(2015). Let $u(t,x), e(t,x)$ and $\vec{F}(t,x)$ denote the body temparature, internal energy and flux density, reps. the the relations

$$\begin{aligned} \partial_t e(t,x) &= -div\,\vec{F}(t,x) \\ e(t,x) &= \beta u(t,x), \quad \vec{F}(t,x) = -\lambda \nabla u(t,x) \end{aligned} \tag{28}$$

yields the classical heat equation $\beta \partial_t u = \lambda \Delta u$.
According to the law of classical heat equation, the speed of heat flow is infinite. However in real modeling, the propagation speed can be finite because the heat flow can be disrupted by the response of the material.

In a material with thermal memory

$$e(t,x) = \bar{\beta}u(t,x) + \int_0^t n(t-s)u(s,x)ds$$

holds with some appropriate constant $\bar{\beta}$ and kernel n. Typically $n(t) = \Gamma(1-\beta)^{-1}t^{-\beta_1}$. The convolution implies that the nearer past affects the present more! If in addition the internal energy also depends on past random effects, then

$$e(t,x) = \bar{\beta}u(t,x) + \int_0^t n(t-s)u(s,x)ds + \int_0^t l(t-s)h(s,u)\ \dot{W}\ (s,x)ds \tag{29}$$

Where $\dot{W}$ is the space time white noise, modeling the random effects. Take $\bar{\beta} = 0$, $l(t) = \Gamma(2-\beta_2)^{-1}t^{1-\beta_2}$, then after differentiation (29) gives $\partial_t^{\beta_1} u = div\ \vec{F} + \frac{1}{\Gamma(1-\beta_2)}\int_0^t (t-s)^{-\beta_2}h(s,u(s,x))\ \dot{W}\ (s,x)ds$

Walsh-Dalang Integral

Need to make sense of the stochastic integral in the mild solution (13). We use the Brownian Filtration $\{\mathcal{F}_t\}$ and the Walsh-Dalang integrals:

- $(t, x) \to \Phi_t(x)$ is an elementary random field when $\exists 0 \leq a < b$ and an $\mathcal{F}_a$-meas. $X \in L^2(\Omega)$ and $\phi \in L^2(\mathbb{R}^d)$ such that
$$\Phi_t(x) = X 1_{[a,b]}(t)\phi(x) \quad (t > 0, x \in \mathbb{R}^d).$$
- If $h = h_t(x)$ is non-random and Φ is elementary, then
$$\int h\Phi dW := X \int_{(a,b)\times\mathbb{R}^d} h_t(x)\phi(x)W(dtdx).$$
- The stochastic integral is Wiener's; well defined iff $h_t(x)\phi(x) \in L^2([a, b] \times \mathbb{R}^d)$.
- We have Walsh isometry,
$$\mathbb{E}\left(\left|\int h\Phi dW\right|^2\right) = \int_0^\infty ds \int_{\mathbb{R}^d} dy[h_s(y)]^2\mathbb{E}(|\Phi_s(y)|^2)$$

Picard iteration

Define $u_t^{(0)}(x) := u_0(x)$, and iteratively define $u_t^{(n+1)}$ from $u_t^{(n)}$ as follows:

$$u_t^{(n+1)}(x) := (G_t * u_0)(x) + \lambda \int_{(0,t)\times\mathbb{R}^d} G(t-r, x-y)\sigma(u^{(n)}(r,y))W(drdy) \tag{30}$$

for all $n \geq 0$, $t > 0$, and $x \in \mathbb{R}^d$. Moreover, we set $u_0^{(k)}(x) := u_0(x)$ for every $k \geq 1$ and $x \in \mathbb{R}^d$.

Proposition (A stochastic Young inequality)

For all $\gamma > 0$, $k \in [2, \infty)$, $d < \min\{2, \beta^{-1}\}\alpha$, *and* $\Phi \in \mathcal{L}^{\gamma,2}$,

$$\mathcal{N}_{\gamma,k}(G \circledast \Phi) \leq c_0 k^{1/2} \cdot \mathcal{N}_{\gamma,k}(\Phi).$$

$$\begin{aligned}
||(G \circledast \Phi)_t(x)||_k^2 &\leq 4k \int_0^t ds \int_{\mathbb{R}^d} dy [G(t-s, y-x)]^2 ||\Phi(y)||_k^2 \\
&\leq 4k[\mathcal{N}_{\gamma,k}(\Phi)]^2 \int_0^t e^{2\gamma s} ds \int_{\mathbb{R}^d} [G(t-s, y-x)]^2 dy \\
&= 4kC^*[\mathcal{N}_{\gamma,k}(\Phi)]^2 \int_0^t e^{2\gamma s} (t-s)^{-\beta d/\alpha} ds \\
&= 4kC^*[\mathcal{N}_{\gamma,k}(\Phi)]^2 e^{2\gamma t} \int_0^t e^{-2\gamma u} u^{-\beta d/\alpha} du \\
&\leq kC_{\alpha,\beta,d}[\mathcal{N}_{\gamma,k}(\Phi)]^2 e^{2\gamma t} (\gamma)^{-(1-\beta d/\alpha)}.
\end{aligned}$$

Continuous time random walks (CTRW)

Particle jump random walk has scaling limit $c^{-1/2}S([ct]) \implies W(t)$.
Number of jumps has scaling limit $c^{-\beta}N(ct) \implies Q(t)$.
CTRW is a random walk subordinated to (a renewal process) $N(t)$

$$S(N(t)) = X_1 + X_2 + \cdots + X_{N(t)}$$

CTRW scaling limit is a subordinated process:

$$\begin{aligned} c^{-\beta/2}S(N(ct)) &= (c^{\beta})^{-1/2}S(c^{\beta}\cdot c^{-\beta}N(ct)) \\ &\approx (c^{\beta})^{-1/2}S(c^{\beta}Q(t)) \implies W(Q(t)). \end{aligned}$$

Figure: Typical sample path of the iterated process $W(Q(t))$. Here $W(t)$ is a Brownian motion and $Q(t)$ is the inverse of a $\beta = 0.8$-stable subordinator. Graph has dimension $1 + \beta/2 = 1 + 0.4$. The limit process retains long resting times

Power law waiting times

- Wait between solar flares $1 < \beta < 2$
- Wait between raindrops $\beta = 0.68$
- Wait between money transactions $\beta = 0.6$
- Wait between emails $\beta \approx 1.0$
- Wait between doctor visits $\beta \approx 1.4$
- Wait between earthquakes $\beta = 1.6$
- Wait between trades of German bond futures $\beta \approx 0.95$
- Wait between Irish stock trades $\beta = 0.4$ (truncated)