On the maximum of the Riemann zeta function on random intervals of the critical line

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Presentation of the setting

- The Riemann zeta function is the analytic continuation of the function defined by
$$\zeta(s) = \sum_{n \geq 1} n^{-s}$$
for $\Re(s) > 1$.
- It is meromorphic in the whole complex plane, with a simple pole at $s = 1$, and infinitely many zeros.
- The zeros are the even negative integers (called trivial zeros), and infinitely many zeros whose real part is in $(0, 1)$ (called non-trivial zeros).
- The non-trivial zeros are symmetrically distributed with respect to the axis $\Re(s) = 1/2$. The Riemann hypothesis states that they are all on the critical line.

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- The behavior of ζ on the critical line $\Re(s) = 1/2$ has been intensively studied, and in particular the order of magnitude of its growth when $t \to \infty$. The Riemann hypothesis implies the so-called *Lindelöf hypothesis*, stating that for any $\varepsilon > 0$, $|\zeta(1/2 + it)| = O((1 + |t|)^{\varepsilon})$.
- The Lindelöf hypothesis is still open today, the best result in this direction is due to Bourgain, who has shown that the bound is true for all $\varepsilon > 13/84$.
- Under the Riemann hypothesis, it is known (in particular from results by Montgomery, Ramachandra, Soundararajan, Titshmarch) that for $t \geq 10$,

$$|\zeta(1/2 + it)| = O\left(e^{\log t/\log\log t}\right), \ |\zeta(1/2 + it)| \neq O\left(e^{\sqrt{\log t/\log\log t}}\right).$$

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- A central limit theorem, due to Selberg, is known for the values of $\log|\zeta|$ on random points of the critical line.
- The average spacing of the zeros whose imaginary part is arount t has order of magnitude $2\pi/\log t$.
- By using Selberg's central limit theorem, and reasoning as if the values of ζ at points distant from $2\pi/\log t$ behave independently, one gets a heuristic
$$\max_{t \in [0,T]} \log|\zeta(1/2+it)| \sim \sqrt{c \log T \log\log T},$$
where $c = 1$.
- By doing a more careful analysis, to take into account the dependence between different values of ζ, Farmer, Gonek and Hughes have conjectured that we have the same asymptotics with $c = 1/2$.

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- Fyodorov, Hiary and Keating have made a very precise conjecture about the order of magnitude of the maximum of $\log|\zeta|$ on random intervals of the critical line with fixed size.
- Their conjecture is related to the fact that $\log|\zeta|$ has correlations which depend on the logarithm of the distance between the points, as proven by Bourgade.
- The conjecture can be stated as follows: for $h > 0$ fixed, $T > 0$, U uniformly distributed on $[0, 1]$,

$$\max_{t \in [UT-h, UT+h]} \log|\zeta(1/2+it)| - (\log\log T - \frac{3}{4}\log\log\log T) \underset{T\to\infty}{\longrightarrow} K,$$

when K is a random variable.

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- A similar conjecture have been stated for the maximum of the characteristic polynomial of the Circular Unitary Ensemble. They are still open, but successive improvements have been successively obtained by Arguin, Belius and Bourgade, Paquette and Zeitouni, Chhaibi, Madaule and N.
- In November 2016, in the setting of the Riemann function, we have proven the following: for all $\varepsilon > 0$, unconditionally,

$$\max_{t \in [UT-h, UT+h]} \Re \log \zeta(1/2 + it) \leq (1+\varepsilon) \log\log T,$$

and under the Riemann hypothesis,

$$\max_{t \in [UT-h, UT+h]} \Re \log \zeta(1/2 + it) \geq (1-\varepsilon) \log\log T.$$

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- We have proven, under the Riemann hypothesis, the same upper bound and the same lower bound for the imaginary part of $\log \zeta$. This gives information on the fluctuations of the distribution of the zeros of ζ on random intervals of the critical line.
- In December 2016, Arguin, Belius, Bourgade, Raziwill, Soundararajan, managed to get rid of the Riemann hypothesis for the lower bound on $\Re \log \zeta$.
- However, it is not known for the moment if the results on $\Im \log \zeta$ occur unconditionally.
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Averaging of $\log|\zeta|$ on the critical line

- For $\Re(s) > 1$, we have

$$\log \zeta(s) = \sum_{n \geq 1} \ell(n) n^{-s}$$

where $\ell(n) = 1/k$ if n is the k-th power of a prime and $\ell(n) = 0$ otherwise.

- If φ is a nonnegative function with integral 1, and if $H > 1$, we get

$$\int_{-\infty}^{\infty} \varphi(t) \log \zeta(s + itH^{-1}) dt = \sum_{n \geq 1} \ell(n) n^{-s} \widehat{\varphi}(H^{-1} \log n).$$

- If we take $\widehat{\varphi}$ compactly supported, the last sum is supported in $n \leq e^{O(H)}$. By analytic continuation arguments, one shows that *under the Riemann hypothesis*, for H sufficiently small with respect to the argument of s, the equality remains true up to a bounded error term, when $\Re(s) \in [1/2, 1)$.

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- With high probability, it is possible to take, for some fixed $\delta \in (0, 1/2)$, $H = \lfloor (\log T)^{1-\delta} \rfloor$, if $s = 1/2 + it$, $t \in [UT - h, UT + h]$.
- Averaging $\log|\zeta|$ tends to smooth its behavior, and then to decrease its maximum.
- It is possible to show that one can replace the smooth cutoff of the sum with $\widehat{\varphi}$ by a sharp cutoff, and remove the powers of primes with exponents at least 2, by doing an error $o(\log \log T)$ on the maximum with high probability.

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- Replacing a maximum on $[UT-h, UT+h]$ by a maximum on a finite subset decreases it.
- Because of these considerations, it is enough to prove the following result, in order to get the lower bound in our main theorem: with high probability, the supremum of

$$\Im \sum_{p \in \mathcal{P}, p \leq e^H} p^{-1/2 - i(UT - h + 2hk/H)},$$

on $k \in \{0, 1, \dots, H-1\}$ is larger than $(1-\varepsilon)\log\log T$, if δ is taken sufficiently small depending on ε.

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Comparison with Gaussian variables

- We cut the previous sum into K parts with approximately equal variance, for K depending on δ and ε, but not on T:

$$S(k,m) = \Im \sum_{p \in \mathcal{P}, e^{e^{m \log H/K}} < p \leq e^{e^{(m+1)\log H/K}}} p^{-1/2 - i(UT - h + 2hk/H)}$$

- For techincal reasons, we also consider a truncation $S_0(k,m)$ at level $(\log T)^{\delta/3}$.
- It is natural to compare the phases p^{-iUT} with i.i.d. variables X_p on the unit circle.

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- It is natural to compare the phases p^{-iUT} with i.i.d. variables X_p on the unit circle.

- We then consider

$$V(k,m) = \Im \sum_{p \in \mathcal{P}, e^{e^{m \log H/K}} < p \leq e^{e^{(m+1)\log H/K}}} X_p p^{-1/2 - i(-h + 2hk/H)}$$

- By bounding the differences of moments in a suitable way, it is possible to show that the joint Fourier transform of $(S_0(k,m), S_0(\ell,m))_{1 \leq m \leq K-1}$ is close to the corresponding Fourier transform with V instead of S_0, for all k, ℓ and at frequencies which are not too large.
- In the moment computations, it is crucial that all products of primes which are involved are much smaller than T, which is guaranteed by the choice of H.

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- In the moment computations, it is crucial that all products of primes which are involved are much smaller than T, which is guaranteed by the choice of H.

- Using the independence of the X_p's, we then compare the Fourier transform of the variable V to the Fourier transform of Gaussian variables G with the same covariance structure.
- This covariance structure satisfies the following:

$$\mathbb{E}[G(k,m)^2] = \frac{\log H}{2K}, \ \mathbb{E}[G(k,m)G(\ell,m)] = O\left(\frac{H^{1-(m/K)}}{|k-\ell|}\right),$$

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the variables with different indices m being independent.

Sketch of proof of the lower bound

- We choose a suitable level $x > 0$ and a nonnegative smooth function φ, with compact support included in $\mathbb{R}_+$.
- We then consider the following variable:

$$J = \sum_{k=0}^{H-1} \prod_{m=1}^{K-1} \varphi(S_0(k,m) - x)$$

- If $J > 0$, then there exists k such that $S_0(k,m) \geq x$, and then $S(k,m) \geq x$ for all m between 1 and $K-1$. Hence, the sum $S(k,1) + \cdots + S(k,K-1)$ is at least $x(K-1)$. Choosing x appropriately, and using a quite rough bound for $S(k,0)$ deduced from an estimate on its moments, we can deduce a suitable lower bound for the sum we have to deal with.

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- Paley-Zygmund's inequality (which is a direct consequence of Cauchy-Schwarz in this setting) implies that

$$\mathbb{P}(J > 0) \geq \frac{(\mathbb{E}[J])^2}{\mathbb{E}[J^2]}.$$

- Hence it is enough to show:

$$\mathbb{E}[J^2] \leq (\mathbb{E}[J])^2 (1 + o(1)),$$

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- Such bound is obtained by comparison of

$$\mathbb{E}\left[\prod_{m=1}^{K-1} \varphi(S_0(k,m)-x)\varphi(S_0(\ell,m)-x)\right]$$

and

$$\mathbb{E}\left[\prod_{m=1}^{K-1} \varphi(S_0(k,m)-x)\right]\mathbb{E}\left[\prod_{m=1}^{K-1} \varphi(S_0(\ell,m)-x)\right].$$

- By inverting the Fourier transform of the variables and using the fact that they are close to each other (in a sense which is made precise), we are able to prove that the quantities just above can be replaced by the similar quantities with S_0 replaced by G, with an acceptable error term.
- The, we use classical Gaussian computations to get estimates on the quantities involving G.

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Some improvement of our result?

- Removing the Riemann hypothesis? For the real part, it has been done, but the authors use ζ instead of $\log \zeta$ at some steps of their reasoning, which cannot be directly transposed to the imaginary part.
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- We need to take H closer to T (with $H = \lfloor (\log T)^{1-\delta} \rfloor$, we already lose a constant times $\log \log T$ in the lower bound we can obtain).
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Thank you for your attention!