Stochastic reaction-diffusion equations on graphs

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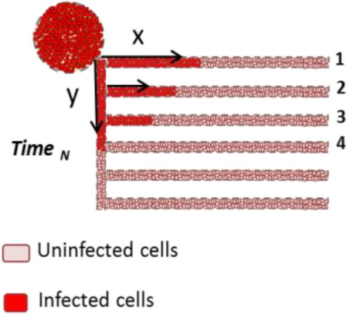
How do spatial structure affects the dynamics, competition outcome and genealogies of interacting populations?



[Inankur and Yin 2015]

Our approach: **develop and analyze novel RDE and SPDE on graphs**

1. SPDE on manifolds [Tindel and Viens 1999, 2002]
2. SPDE on graphs first appeared in [Cerrai and Freidlin 2014, 2016]

$$\begin{cases} \partial_t u = \alpha\, \Delta u + b(u) + \sigma(u)\dot{W} & \text{on } \mathring{G} \\ \nabla_{out} u \cdot \vec{\alpha} = 0 & \text{on } V \end{cases}$$

as the limit of a two dimensional SPDE.

(I) We introduce **more general** SPDE on graphs

$$\begin{cases} \partial_t u = \alpha\, \Delta u + b(u) + \sigma(u)\, \dot{W} & \text{on } \mathring{G} \\ \nabla_{out} u \cdot \vec{\alpha} = -\hat{\beta}(u) & \text{on } V \end{cases}$$

(II) We obtain **the first scaling limit results** which rigorously connect individual based models to both deterministic and noisy RDE on general metric graphs.

SPDE ON GRAPHS

Precisely,

$$\begin{cases} \partial_t u = \alpha(x)\,\Delta u(t,x) + b\big(x,\ u(t,x)\big) + \sigma\big(x,\ u(t,x)\big)\,\dot{W} & \text{for } x \in \mathring{G} \\ \nabla_{out} u \cdot \vec{\alpha} = -\hat{\beta}(v,\ u(t,v)) & \text{for } v \in V \end{cases}$$

is the shorthand of

$$\begin{aligned} u_t(x) = P_t u_0(x) &+ \int_0^t P_{t-s}\big(b(\cdot, u_s)\big)(x)\,ds \\ &+ \int_{[0,t]\times G} p(t-s,x,y)\sigma(y,u_s(y))\,dW(s,y) \\ &+ \int_0^t \sum_{v\in V} p(t-s,x,v)\,\hat{\beta}(v,u_s(v))\,ds, \end{aligned}$$

where $\{P_t\}_{t\geq 0}$ is the $\mathcal{C}_\infty(G)$-semigroup and $p(t,x,y)$ the transition density for the symmetric α-diffusion on G [Freidlin and Wentzell 1993, Freidlin and Sheu 2000].

SPDE on graphs

Example (Branching random walks)

$$\begin{cases} \partial_t u = \alpha_e \Delta u + \beta_e u + \sqrt{4\gamma_e u}\,\dot{W} & \text{on } \mathring{e} \\ \nabla_{out} u \cdot \vec{\alpha} = -\hat{\beta}(u) & \text{on } V. \end{cases}$$

Example (Contact process)

$$\begin{cases} \partial_t u = \alpha_e \Delta u + \beta_e u - \delta_e u^2 + \sqrt{\gamma_e u}\,\dot{W} & \text{on } \mathring{e} \\ \nabla_{out} u \cdot \vec{\alpha} = -\hat{\beta}(u) & \text{on } V. \end{cases}$$

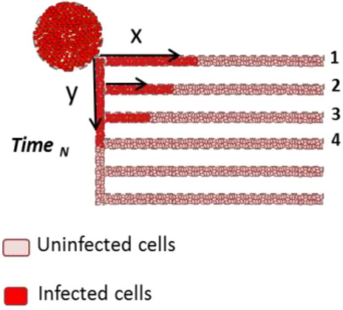
Example (Ginzburg-Landau equation)

$$\begin{cases} \partial_t u = \alpha_e \Delta u + \beta_e u - \delta_e u^3 + \sqrt{\gamma_e}\,\dot{W} & \text{on } \mathring{e} \\ \nabla_{out} u \cdot \vec{\alpha} = -\hat{\beta}(u) & \text{on } V. \end{cases}$$

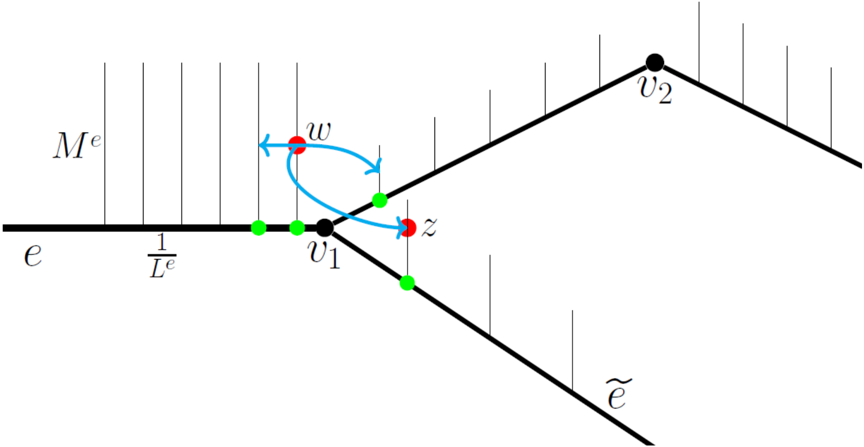
SPDE ON GRAPHS

EXAMPLE (BVM)

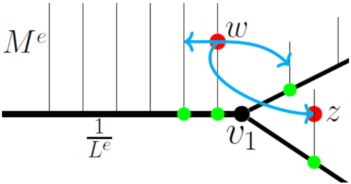
$$\begin{cases} \partial_t u &= \alpha_e \, \Delta u + \beta_e \, u(1-u) + \sqrt{\gamma_e \, u(1-u)}\,\dot{W} & \text{on } \mathring{e} \\ \nabla_{out} u \cdot \vec{\alpha} &= -\hat{\beta}\, u(1-u) & \text{on } V. \end{cases}$$



Individual based modeling



- $1/L^e$ as the diameter of a biological cell,
- M^e as the capacity of each cell.



CASE (I) if w is on e^n and is *not* adjacent to any vertex, then

$$a^{z,w} = \frac{\alpha_e (L^e)^2}{M^e} \quad \text{and} \quad b^{z,w} = \frac{\beta_e}{M^e}$$

CASE (II) if w is on e^n and is adjacent to $v \in V$ while z is on $\tilde{e}$,

$$a^{z,w} = \frac{L^e \, C_{e,\tilde{e}}}{M^e} \quad \text{and} \quad b^{z,w} = b^{\tilde{e},e}$$

With rate $a^{z,w}$, z is replaced by an offspring of w. With rate $b^{z,w}$, z is replaced by an offspring of w **only if** w **has type 1**.

Theorem (F, 2017)

Given $\hat{\beta}(v)$ *for all vertex* $v \in V$ *and a triple* $(\alpha_e, \beta_e, \gamma_e)$ *for each edge* e. *Suppose*

1. $L^e/M^e \to \gamma_e/4\alpha_e$ *and* $L^e \to \infty$ *for all* e,
2. *All* L^e *are comparable,*
3. $\sum_{e\in E(v)} \sum_{\tilde{e}\in E(v)} \frac{b^{\tilde{e},e}\, M^{\tilde{e}}}{L^e} \to \hat{\beta}(v)$.

Then the approximate density process converges in distribution in $\mathcal{D}([0,\infty), \mathcal{C}_{[0,1]}(G))$ *to a continuous* $\mathcal{C}_{[0,1]}(G)$ *valued process* $(u_t)_{t\geq 0}$ *which is the weak solution to*

$$\begin{cases} \partial_t u = \alpha_e\, \Delta u + \beta_e\, u(1-u) + \sqrt{\gamma_e\, u(1-u)}\,\dot{W} & \text{on } \mathring{e} \\ \nabla_{out} u \cdot \vec{\alpha} = -\hat{\beta}\, u(1-u) & \text{on } V. \end{cases}$$

We have two more approximating schemes:
Interacting SDEs and **Interacting random walks**.

Proof outline:

Step 1) $\{u^n\}_{n\geq 1}$ is $\mathcal{C}$-tight in $\mathcal{D}([0,T], \mathcal{C}_{[0,1]}(G))$ for every $T > 0$.

Step 2) Any sub-sequential limit u is a weak solution.

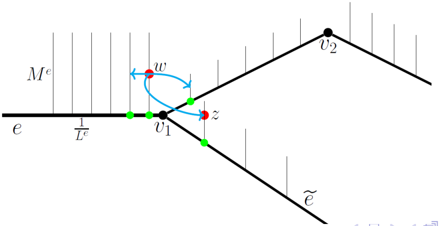
Step 3) Weak uniqueness for the SPDE holds

New challenges:

(A) Interactions near vertex singularities in relation to $\{M^e\}$ and $\{L^e\}$.

(B) Weak uniqueness via a new duality

(C) Uniform heat kernel estimates for random walks and diffusions on G



Non-uniqueness of SPDE

Theorem (Mueller, Mytnik and Perkins 2014)

If $0 < \gamma < 3/4$, then uniqueness in law and pathwise uniqueness **fail** *for*

$$\partial_t u(t,x) = \frac{1}{2}\Delta u(t,x) + |u(t,x)|^{\gamma}\,\dot{W}(t,x), \qquad u(0,x) = 0$$

on the space of $\mathcal{C}(\mathbb{R})$-valued adapted processes.

Open questions:

- strong uniqueness for case $\gamma = 3/4$.
- strong uniqueness for stochastic FKPP on $\mathcal{C}_+(\mathbb{R})$-valued processes

Dual process

$$
\begin{cases} \partial_t u = \alpha\,\Delta u + \beta\, u(1-u) + \sqrt{\gamma\, u(1-u)}\,\dot{W} & \text{on } \mathring{G} \\ \nabla_{out} u \cdot \vec{\alpha} = -\hat{\beta}\, u(1-u) & \text{on } V \end{cases} \tag{1.1}
$$

Lemma (F, 2017)

Suppose α, β, γ/α, $\hat{\beta}$ are nonnegative, bounded and continuous. Then (1.1) **has a dual process** *which is a branching coalescing α-diffusions on G*

- *branching for a particle X_t occurs at rate*
$$\beta(X_t)dt + \hat{\beta}(X_t)dL_t^V$$
- *two particles X_t, Y_t coalesce at rate*
$$\frac{\gamma(X_t)}{\alpha(X_t)}\, dL_t^{(X,Y)}.$$

Corollary: **Weak uniqueness** of SPDE (1.1) holds on $\mathcal{C}_{[0,1]}(G)$. Useful to the study of time-asymptotic properties.

Thank you!