The expected Euler characteristic approximation for Gaussian vector fields

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Let $X = \{X(t), t \in T\}$ be a real-valued Gaussian random field, where T is the parameter set. For $u > 0$, how to evaluate the excursion probability

$$\mathbb{P}\left\{\sup_{t\in T} X(t) \geq u\right\} =?$$

Asymptotics for Gaussian fields

- Double sum method [Pickands (1969), Piterbarg (1996), Chan and Lai (2006), etc.]
- Tube method [Sun (1993)]
- Rice method: the expected number of local maxima. See Piterbarg (1996), Adler (2000), Azaïs and Wschebor (2008), etc.
- Euler heuristic: the expected Euler characteristic of the excursion set. See Adler (1981, 2000), Worsley (1995), Taylor, Takemura and Adler (2005), Adler and Taylor (2007), etc.

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Euler characteristic method

- Let $A_u = \{t \in T : X(t) \geq u\}$ be the excursion set.
- Let $\chi(A_u)$ and $\mathbb{E}\{\chi(A_u)\}$ be the Euler characteristic and mean Euler characteristic of A_u respectively.
- In one dimension, the Euler characteristic is the number of connected components; and in two dimensions, it is the number of connected components minus the number of holes.

Theorem [Taylor, Takemura and Adler (2005)]

Let $X = \{X(t), t \in T\}$ be a centered smooth Gaussian random field with unit variance. There exists $\alpha > 0$ such that

$$\mathbb{P}\left\{\sup_{t\in T} X(t) \geq u\right\} = \mathbb{E}\{\chi(A_u)\} + o(e^{-\alpha u^2 - u^2/2}), \text{ as } u \to \infty.$$

- $\mathbb{E}\{\chi(A_u)\}$ is computable [Adler and Taylor (2007)],

$$\mathbb{E}\{\chi(A_u)\} = C_0\Psi(u) + \sum_{j=1}^{\dim(T)} C_j u^{j-1} e^{-u^2/2},$$

  where C_j are constants depending on X and T.
- When X is isotropic and $T = [0, L]^N$,

$$\mathbb{E}\{\chi(A_u)\} = \Psi(u) + \sum_{j=1}^{N} \frac{\binom{N}{j} L^j \lambda^{j/2}}{(2\pi)^{(j+1)/2}} H_{j-1}(u) e^{-u^2/2},$$

  where $\lambda = \text{Var}\left(\frac{\partial X(t)}{\partial t_i}\right)$ and $H_{j-1}(u)$ are Hermite polynomials.

Let $\{(X(t), Y(s)) : t \in T, s \in S\}$ be an $\mathbb{R}^2$-valued, centered, unit-variance smooth Gaussian vector field, where T and S are rectangles in $\mathbb{R}^N$. For $u > 0$, how to evaluate the excursion probability

$$\mathbb{P}\left\{\sup_{t\in T} X(t) \geq u, \sup_{s\in S} Y(s) \geq u\right\} =?$$

- Consider rectangles $T = \prod_{i=1}^{N}[a_i, b_i]$ and $S = \prod_{i=1}^{N}[a'_i, b'_i]$.

Definition

A face J of dimension k, is defined by fixing a subset $\sigma(J) \subset \{1, \cdots, N\}$ of size k and a subset $\varepsilon(J) = \{\varepsilon_j, j \notin \sigma(J)\} \subset \{0,1\}^{N-k}$ of size $N-k$, so that

$$J = \{t \in T : a_j < t_j < b_j \text{ if } j \in \sigma(J), \\ t_j = (1-\varepsilon_j)a_j + \varepsilon_j b_j \text{ if } j \notin \sigma(J)\}.$$

- Let $\partial_k T$ be the collection of faces of dimension k in T, then $\overset{\circ}{T} = \partial_N T$, $\partial T = \cup_{k=0}^{N-1} \cup_{J \in \partial_k T} J$ and $T = \cup_{k=0}^{N} \partial_k T = \cup_{k=0}^{N} \cup_{J \in \partial_k T} J$.
- Similarly, let $\partial_l S$ be the collection of faces of dimension l in S, then $\overset{\circ}{S} = \partial_N S$, $\partial S = \cup_{l=0}^{N-1} \cup_{L \in \partial_l S} L$ and $S = \cup_{l=0}^{N} \partial_l S = \bigcup_{l=0}^{N} \cup_{L \in \partial_l S} L$.

Some notations

For a function $f(\cdot) \in C^2(\mathbb{R}^N)$ and $t \in \mathbb{R}^N$, let

$$f_i(t) = \frac{\partial f(t)}{\partial t_i}, \quad f_{ij}(t) = \frac{\partial^2 f(t)}{\partial t_i \partial t_j}, \quad \forall i, j = 1, \ldots, N,$$
$$\nabla f(t) = (f_1(t), \ldots, f_N(t))^T, \quad \nabla^2 f(t) = (f_{ij}(t))_{i,j=1,\ldots,N}.$$

For each $t \in J \in \partial_k T$ and $s \in L \in \partial_l S$, let

$$\nabla X_{|J}(t) = (X_{i_1}(t), \ldots, X_{i_k}(t))^T_{i_1,\ldots,i_k \in \sigma(J)}, \quad \nabla^2 X_{|J}(t) = (X_{mn}(t))_{m,n \in \sigma(J)},$$
$$\nabla Y_{|L}(s) = (Y_{i_1}(s), \ldots, Y_{i_l}(s))^T_{i_1,\ldots,i_l \in \sigma(L)}, \quad \nabla^2 Y_{|L}(s) = (Y_{mn}(s))_{m,n \in \sigma(L)}.$$

Define the excursion sets

$$A_u(X,T) = \{t \in T : X(t) \geq u\},$$
$$A_u(Y,S) = \{s \in S : Y(s) \geq u\},$$
$$A_u(X,T) \times A_u(Y,S) = \{(t,s) \in T \times S : X(t) \geq u, Y(s) \geq u\}.$$

For each $J \in \partial_k T$ and $L \in \partial_l S$, define the number of extended outward critical point of index i above level u as

$$\mu_i(X,J) := \#\{t \in J : X(t) \geq u, \nabla X_{|J}(t) = 0, \text{index}(\nabla^2 X_{|J}(t)) = i,$$
$$\varepsilon_j^* X_j(t) \geq 0 \text{ for all } j \notin \sigma(J)\},$$
$$\mu_i(Y,L) := \#\{s \in L : Y(s) \geq u, \nabla Y_{|L}(s) = 0, \text{index}(\nabla^2 Y_{|L}(s)) = i,$$
$$\varepsilon_j^* Y_j(s) \geq 0 \text{ for all } j \notin \sigma(L)\},$$

where where $\varepsilon_j^* = 2\varepsilon_j - 1$ and the index of a matrix is defined as the number of its negative eigenvalues.

Morse theorem

It follows from the Morse theorem [see Adler and Taylor (2007)] that the Euler characteristic of the excursion set can be represented as

$$\chi(A_u(X,T)) = \sum_{k=0}^{N} \sum_{J\in\partial_k T} (-1)^k \sum_{i=0}^{k} (-1)^i \mu_i(X,J),$$

$$\chi(A_u(Y,S)) = \sum_{l=0}^{N} \sum_{L\in\partial_l S} (-1)^l \sum_{i=0}^{l} (-1)^i \mu_i(Y,L).$$

Moreover,

$$\chi(A_u(X,T)\times A_u(Y,S)) = \chi(A_u(X,T))\times\chi(A_u(Y,S))$$

$$= \sum_{k,l=0}^{N} \sum_{J\in\partial_k T, L\in\partial_l S} (-1)^{k+l} \left(\sum_{i=0}^{k} (-1)^i \mu_i(X,J)\right)\left(\sum_{j=0}^{l} (-1)^j \mu_j(Y,L)\right).$$

Expected Euler characteristic

Therefore, we have

$$
\begin{aligned}
&\mathbb{E}\{\chi(A_u(X,T) \times A_u(Y,S))\} \\
&= \sum_{k,l=0}^{N} \sum_{J\in\partial_k T, L\in\partial_l S} (-1)^{k+l}\mathbb{E}\left\{\sum_{i=0}^{k}\sum_{j=0}^{l}(-1)^{i+j}\mu_i(X,J)\mu_j(Y,L)\right\}
\end{aligned}
$$

For each $J \in \partial_k T$ and $L \in \partial_l S$, define the number of extended outward maxima above level u as

$$M_u^E(X,J) := \#\{t \in J : X(t) \geq u, \nabla X_{|J}(t) = 0, \text{index}(\nabla^2 X_{|J}(t)) = k, \varepsilon_j^* X_j(t) \geq 0 \text{ for all } j \notin \sigma(J)\},$$

$$M_u^E(Y,L) := \#\{s \in L : Y(s) \geq u, \nabla Y_{|L}(s) = 0, \text{index}(\nabla^2 Y_{|L}(s)) = l, \varepsilon_j^* Y_j(s) \geq 0 \text{ for all } j \notin \sigma(L)\}.$$

Example: Let $T = [0,1] = \{0\} \cup \{1\} \cup (0,1)$, then with probability 1,

$$\left\{\sup_{t\in T} X(t) \geq u\right\} = \{X(0) \geq u, X'(0) \leq 0\} \cup \{X(1) \geq u, X'(1) \geq 0\}$$
$$\cup \{\exists t \in (0,1) \text{ s.t. } X(t) \geq u, X'(t) = 0, X''(t) < 0\}.$$

- For Gaussian fields, since $T = \cup_{k=0}^{N} \cup_{J\in\partial_k T} J$, one can show that

$$\left\{\sup_{t\in T} X(t) \geq u\right\} = \bigcup_{k=0}^{N} \bigcup_{J\in\partial_k T} \{M_u^E(J) \geq 1\} \quad \text{a.s.}$$

- For Gaussian vector fields, since $T = \cup_{k=0}^{N} \cup_{J\in\partial_k T} J$ and $S = \bigcup_{l=0}^{N} \cup_{L\in\partial_l S} L$, one can show that

$$\left\{\sup_{t\in T} X(t) \geq u, \sup_{s\in S} Y(s) \geq u\right\}$$
$$= \bigcup_{k,l=0}^{N} \bigcup_{J\in\partial_k T, L\in\partial_l S} \{M_u^E(X,J) \geq 1, M_u^E(Y,L) \geq 1\} \text{ a.s.}$$

Therefore, we obtain the upper bound for the excursion probability

$$\begin{aligned}
&\mathbb{P}\left\{\sup_{t\in T} X(t) \geq u, \sup_{s\in S} Y(s) \geq u\right\} \\
&\leq \sum_{k,l=0}^{N} \sum_{J\in\partial_k T, L\in\partial_l S} \mathbb{P}\{M_u^E(X,J) \geq 1, M_u^E(Y,L) \geq 1\} \\
&\leq \sum_{k,l=0}^{N} \sum_{J\in\partial_k T, L\in\partial_l S} \mathbb{E}\{M_u^E(X,J) M_u^E(Y,L)\}.
\end{aligned}$$

On the other hand, notice that

$$
\begin{aligned}
&\mathbb{E}\{M_u^E(X,J)M_u^E(Y,L)\} - \mathbb{P}\{M_u^E(X,J) \geq 1, M_u^E(Y,L) \geq 1\} \\
&= \sum_{i,j=1}^{\infty} (ij-1)\mathbb{P}\{M_u^E(X,J) = i, M_u^E(Y,L) = j\} \\
&\leq \sum_{i,j=1}^{\infty} [i(i-1)j + j(j-1)i]\mathbb{P}\{M_u^E(X,J) = i, M_u^E(Y,L) = j\} \\
&= \mathbb{E}\{M_u^E(X,J)[M_u^E(X,J)-1]M_u^E(Y,L)\} \\
&\quad + \mathbb{E}\{M_u^E(Y,L)[M_u^E(Y,L)-1]M_u^E(X,J)\}
\end{aligned} \tag{1}
$$

and

$$
\begin{aligned}
&\mathbb{P}\{M_u^E(X,J) \geq 1, M_u^E(Y,L) \geq 1, M_u^E(X,J') \geq 1, M_u^E(Y,L') \geq 1\} \\
&\leq \mathbb{P}\{M_u^E(X,J) \geq 1, M_u^E(Y,L) \geq 1, M_u^E(Y,L') \geq 1\} \\
&\leq \mathbb{E}\{M_u^E(X,J)M_u^E(Y,L)M_u^E(Y,L')\}.
\end{aligned} \tag{2}
$$

By the Bonferroni inequality, (1) and (2), we obtain the lower bound for the excursion probability

$$\begin{aligned}
&\mathbb{P}\left\{\sup_{t\in T} X(t) \geq u, \sup_{s\in S} Y(s) \geq u\right\} \\
&\geq \sum_{k,l=0}^{N} \sum_{J\in\partial_k T, L\in\partial_l S} \Big\{\mathbb{E}\{M_u^E(X,J)M_u^E(Y,L)\} - \mathbb{E}\{M_u^E(X,J) \\
&\quad \times [M_u^E(X,J)-1]M_u^E(Y,L)\} - \mathbb{E}\{M_u^E(Y,L)[M_u^E(Y,L)-1]M_u^E(X,J)\}\Big\} \\
&\quad - \sum_{k,k',l=0}^{N} \sum_{\substack{J\in\partial_k T, L\in\partial_l S \\ J'\in\partial_{k'} T, J\neq J'}} \mathbb{E}\{M_u^E(X,J)M_u^E(X,J')M_u^E(Y,L)\} \\
&\quad - C_N \sum_{k,l,l'=0}^{N} \sum_{\substack{J\in\partial_k T, L\in\partial_l S \\ L'\in\partial_{l'} S, L\neq L'}} \mathbb{E}\{M_u^E(X,J)M_u^E(Y,L)M_u^E(Y,L')\},
\end{aligned}$$

where C_N is a constant depending only on N.

Let

$$r(t,s) = \mathbb{E}\{X(t)Y(s)\}, \quad \rho = \sup_{t\in T, s\in S} \mathbb{E}\{X(t)Y(s)\}.$$

- We call a function $h(u)$ *super-exponentially small* [when compared with the excursion probability $\mathbb{P}\{\sup_{t\in T} X(t) \geq u, \sup_{s\in S} Y(s) \geq u\}$], if there exists a constant $\alpha > 0$ such that $h(u) = o(e^{-\alpha u^2 - u^2/(1+\rho)})$ as $u \to \infty$.
- The sketch of proof consists of the following two steps: (i) all terms, except for those in the upper bound, are super-exponentially small; (ii) the difference between the upper bound and the expected Euler characteristic of the excursion set is also super-exponentially small.

Lemma Under certain smoothness and regularity conditions, there exists some $\alpha > 0$ such that

$$\mathbb{P}\left\{\sup_{t\in T} X(t) \geq u, \sup_{s\in S} Y(s) \geq u\right\}$$
$$= \sum_{k,l=0}^{N} \sum_{J\in\partial_k T, L\in\partial_l S} \mathbb{E}\{M_u^E(X,J)M_u^E(Y,L)\} + o\left(\exp\left\{-\frac{u^2}{1+\rho} - \alpha u^2\right\}\right)$$

and

$$\sum_{k,l=0}^{N} \sum_{J\in\partial_k T, L\in\partial_l S} \mathbb{E}\{M_u^E(X,J)M_u^E(Y,L)\}$$
$$= \mathbb{E}\{\chi(A_u(X,T) \times A_u(Y,S))\} + o\left(\exp\left\{-\frac{u^2}{1+\rho} - \alpha u^2\right\}\right).$$

Expected Euler characteristic approximation

Theorem Let $\{(X(t), Y(s)) : t \in T, s \in S\}$ be an $\mathbb{R}^2$-valued, centered, unit-variance Gaussian vector field. Then under certain smoothness and regularity conditions, there exists $\alpha > 0$ such that

$$\mathbb{P}\left\{\sup_{t\in T} X(t) \geq u, \sup_{s\in S} Y(s) \geq u\right\}$$
$$= \mathbb{E}\{\chi(A_u(X,T) \times A_u(Y,S))\} + o\left(\exp\left\{-\frac{u^2}{1+\rho} - \alpha u^2\right\}\right).$$

Some remarks

- The expected Euler characteristic approximation $\mathbb{E}\{\chi(A_u(X,T)\times A_u(Y,S))\}$ can be written as an integral by the Kac-Rice formula. However, due to the correlation function between X and Y, it is usually difficult to obtain the exact value. We usually need to apply the Laplace method to approximate the integral.
- The expected Euler characteristic approximation can be extended to general $\mathbb{R}^d$-valued Gaussian vector fields.
- Extensions for more general T and S are possible (such as spheres).

Thank you!