Between-block dependence under long-range dependence

Shuyang Bai*

joint work with Murad S. Taqqu**

University of Georgia* & Boston University**

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Time domain perspective of long-range dependence (LRD)

$\{X_n\}$: stationary time series.

Short-range dependence (SRD):

$$\sum_{k=-\infty}^{\infty} \left|\mathrm{Cov}[X_k, X_0]\right| < \infty, \quad \sum_{k=-\infty}^{\infty} \mathrm{Cov}[X_k, X_0] > 0.$$

Example: ARMA(p,q) models.

Long-range dependence (LRD): ($\sim$: asymptotic equivalence up to a positive constant.)

$$\mathrm{Cov}[X_k, X_0] \sim k^{-\beta}, \text{ as } k \to \infty, \ \beta \in (0, \mathbf{1}).$$

Critical Value: $\beta = \mathbf{1}$.

	SRD	LRD
β	$(1, +\infty)$	$(0, 1)$
$\sum_{k=-\infty}^{+\infty} \left|\mathrm{Cov}[X_k, X_0]\right|$	$< \infty$	$= \infty$
$\sqrt{\mathrm{Var}[X_1 + \ldots + X_n]}$	$\sim n^{1/2}$	$\sim n^{1/2+(1-\beta)/2}$

Frequency domain perspective of LRD

From time-domain perspective, LRD is $\gamma(k) = \mathrm{Cov}[X_k, X_0] \sim k^{-\beta}$, $\beta \in (0,1)$.

We also have a frequency (Fourier)-domain characterization of LRD.

Spectral density (power spectrum):

$$f(\lambda) = \lim_{N\to\infty} \frac{1}{2\pi N} \mathbb{E} \left| \sum_{n=-N}^{N} X(n) e^{-i\lambda n} \right|^2 = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{-in\lambda} \gamma(n), \quad \lambda \in (-\pi, \pi]$$

Interpretation: the long-run average "energy" at frequency λ.

- X_n SRD: $f(\lambda)$ is bounded away from 0 and $+\infty$ and is continuous;
- X_n LRD: $f(\lambda) \sim |\lambda|^{\beta-1}$ near $\lambda = 0$, $\beta \in (0,1)$.

  Overwhelming energy in low frequencies.
- Time and frequency domain perspectives of LRD are *typically* equivalent.

Strong mixing and c-mixing

The α-mixing coefficient between σ-fields $\mathcal{F}$, $\mathcal{G}$:

$$\begin{aligned}\alpha(\mathcal{F},\mathcal{G}) : &= \sup\{|\mathrm{Cov}[1_A, 1_B]| : \ A \in \mathcal{F}, B \in \mathcal{G}\} \\ &= \sup\{|P(A \cap B) - P(A)P(B)| : \ A \in \mathcal{F}, B \in \mathcal{G}\}.\end{aligned}$$

$(X_n)_{n\in\mathbb{Z}}$ stationary. Let $\mathbf{X}_p^q = (X_p, X_{p+1}, \dots, X_q)$, $-\infty \le p \le q \le +\infty$.

(X_n) is said to be *strongly mixing* (or *α-mixing*), if

$$\alpha_k := \alpha\left(\sigma(\mathbf{X}_{-\infty}^0), \sigma(\mathbf{X}_k^\infty)\right) \to 0 \quad \text{as } k \to \infty.$$

Typically, LRD $\Rightarrow$ strong mixing fails, i.e.,

$$\lim_{k\to\infty} \alpha_k = \lim_{k\to\infty} \lim_{b\to\infty} \alpha_{k,b} > 0. \tag{1}$$

where

$$\alpha_{k,b} := \alpha\left(\sigma(\mathbf{X}_{-b+1}^0), \sigma(\mathbf{X}_k^{k+b-1})\right).$$

On the other hand under LRD, one typically expects

$$\lim_{k\to\infty} \alpha_{k,b} = 0, \qquad \forall \textbf{ fixed } b < \infty, \quad \text{termed } c\textit{-mixing by Doukhan}. \tag{2}$$

A gap between strong mixing (1) and c-mixing (2)! What if $b, k \uparrow +\infty$ together?

Between-block mixing coefficient

Recall: $\alpha_{k,b} := \alpha\left(\sigma(\mathbf{X}_{-b+1}^{0}), \sigma(\mathbf{X}_{k}^{k+b-1})\right)$, non-decreasing in b.

Theorem (Bai & Taqqu (2016))

If (X_n) is LRD Gaussian, $\mathrm{Cov}[X_k, X_0] \sim k^{-\beta}$, $\beta \in (0,1)$, satisfying some additional regularity conditions. Then $\forall \lambda > 0$, $\exists\ 0 < c \leq C$,

$$c\left(\frac{b}{k}\right)^{\beta} \leq \alpha_{k,b} \leq C\left(\frac{b}{k}\right)^{\beta}, \quad \textit{for all } 1 \leq b \leq \lambda k.$$

Corollary

$\alpha_{k,b} \to 0$ as $k, b \uparrow \infty$ iff $b = o(k)$.

Corollary

$\sum_{k=1}^{n} \alpha_{k,b} = o(n)$ as $b \uparrow \infty$ and $b = o(n)$.

Remark (about the 2nd Corollary)

- *Consistency of certain resampling procedures under LRD (Politis & Romano, 1994).*
- *Resampling is important under LRD. E.g. $\frac{1}{n^H}\sum_{i=1}^{[nt]} G(X_i) \Rightarrow Z_{m,H}(t)$, where $Z_{m,H}$ is the Hermite process. Hermite rank m (nuisance parameter) depends on the unobservable G.*

Regularity condition

Precise statement: LRD Gaussian (X_n), $\mathrm{Cov}[X_n, X_0] \sim n^{-\beta}$, $\beta \in (0,1)$. Assume the spectral density of (X_n) is given by

$$f(\lambda) = f_\beta(\lambda) f_0(\lambda),$$

where $f_\beta(\lambda)$ is the FARIMA(0,$d = \frac{1-\beta}{2}$,0) spectrum:

$$f_\beta(\lambda) = |1 - e^{i\lambda}|^{\beta - 1},$$

and $f_0(\lambda)$ satisfies SRD conditions ($\gamma_0(n)$ is the covariance of $f_0(\lambda)$):

(a) $\inf_\lambda f_0(\lambda) > 0$; (b) $\gamma_0(n) = O(n^{-\alpha})$, $\alpha > 1$.

Then $\forall \lambda > 0$, $\exists\ 0 < c \le C$

$$c\left(\frac{b}{k}\right)^\beta \le \alpha_{k,b} \le C\ \left(\frac{b}{k}\right)^\beta + \underbrace{\cancel{O(k^{-\alpha+1})}}_{\text{if } \alpha > 1+\beta}, \quad \text{for all } 1 \le b \le \lambda k.$$

- Time-domain interpretation: Let $d = \frac{1-\beta}{2}$, FARIMA model: $\Delta^d X_n = \epsilon_n$, (ϵ_n) has $f_0(\lambda)$.
- Examples: FARIMA(p,d,q), fractional Gaussian noise $H > 1/2$.
- If $f_\beta(\lambda)$ is absent and $\gamma_0(n) \sim n^{-\alpha}$, $\alpha > 1$ (SRD), then $\alpha_{k,b} \asymp bk^{-\alpha}$ (v.s. LRD $b^\beta k^{-\beta}$).

Strategy of proof

- Gaussian maximal correlation (Kolmogorov and Rozanovm 1960)

$(\mathbf{Z}_1, \mathbf{Z}_2)$ jointly Gaussian. Define the maximal linear (canonical) correlation

$$\rho(\mathbf{Z}_1, \mathbf{Z}_2) := \sup_{\mathbf{a},\mathbf{b}} \mathrm{Corr}\Big(\langle \mathbf{a}, \mathbf{Z}_1\rangle, \langle \mathbf{b}, \mathbf{Z}_2\rangle\Big), \quad \langle \cdot, \cdot \rangle : \text{ Euclidean inner product.}$$

Then

$$\frac{1}{2\pi}\rho(\mathbf{Z}_1, \mathbf{Z}_2) \le \alpha\left(\sigma(\mathbf{Z}_1), \sigma(\mathbf{Z}_2)\right) \le \frac{1}{4}\rho(\mathbf{Z}_1, \mathbf{Z}_2).$$

- Upper bound:

(1) For the FARIMA($0, d = \frac{1-\beta}{2}, 0$) spectrum $f_d(\lambda) = |1 - e^{i\lambda}|^{-2d}$, explore some explicit formulas from linear prediction theory to get $\rho_{k,b} \le c(b/k)^{\beta}$.

(2) Insert the well-behaved SRD $f_0(\lambda)$ via Fourier analysis to get $\rho_{k,b} \le c(b/k)^{\beta} + o_k(1)$.

- Lower bound is easy (recall $\gamma(k) \sim k^{-\beta}$, $0 < \beta < 1$):

$$\begin{aligned}\rho_{k,b} :&= \rho(\mathbf{X}^{0}_{-b+1}, \mathbf{X}_k^{k+b-1}) \ge \mathrm{Corr}[(X_{-b+1} + \ldots + X_0), (X_k + \ldots + X_{k+b-1})] \\ &= \frac{\mathrm{Cov}[(X_{-b+1} + \ldots + X_0), (X_k + \ldots + X_{k+b-1})]}{\mathrm{Var}[X_{-b+1} + \ldots + X_0]} \\ &\gtrsim \frac{1}{b^{-\beta+2}} \sum_{1 \le i,j \le b} \gamma(i - j + k + b - 1) \gtrsim b^{\beta-2} b^2 k^{-\beta} = k^{-\beta} b^{\beta}.\end{aligned}$$

($b^{-\beta+2}$ replaced by b if instead $\beta > 1$.)

Summary

• A bound on dependence between two finite blocks of the LRD time series is derived. Useful for justifying resampling procedures under LRD.

• Reference: On the validity of resampling methods under long memory (with Murad S. Taqqu) (2016) (To appear in *The Annals of Statistics*).

Open Problem

LRD linear processes $X_n = \sum_{j=0}^{\infty} a_j \epsilon_{n-j}$, $a_j \sim j^{-\beta/2-1/2}$, $\beta \in (0,1)$, (ϵ_n) i.i.d. non-Gaussian.

By adapting the arguments of literature on strong mixing of linear processes (given sufficient regularity except Gaussianity),

$$\alpha_{k,b} \leq \alpha(\sigma(\mathbf{X}_{-\infty}^{0}, \mathbf{X}_{k}^{k+b-1})) \leq Ck^{-\beta/2+\epsilon} b, \qquad \text{v.s. the Gaussian case: } k^{-\beta} b^{\beta}.$$