Behavior of the generalized Rosenblatt process at extreme critical exponent values

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(Joint work with Shuyang Bai)

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This talk is based on the paper:

S. Bai and M.S. Taqqu. "Behavior of the generalized Rosenblatt process at extreme critical exponent values". To appear in *The Annals of Probability*, 2016.

Self-similarity

We will focus on self-similar processes with stationary increments

Definition: A process $Z(t), t \geq 0$ is self-similar with parameter $H > 0$ if for any real a the finite-dimensional distributions of $Z(at), t \geq 0$ are the same as those of $a^H Z(t), t \geq 0$.

The self-similar processes we will focus on, are by order of complexity:

- Trivial process $X(t) = Xt$ is self-similar with $H = 1$.
- Brownian motion is self-similar with $H = 1/2$.
- Fractional Brownian motion is self-similar with $0 < H < 1$.
- The Rosenblatt process is self-similar with $1/2 < H < 1$.
- The Generalized Rosenblatt process is self-similar with $1/2 < H < 1$.

The Trivial process

Any right-continuous self-similar process $X(t)$ with $\mathbb{E}|X(t)|^2 < \infty$, stationary increments and $H = 1$ is the trivial process

$$X(t) = tX(1) \quad a.s.$$

Fractional Brownian motion

$B_H(t), \ t \geq 0$, called **Fractional Brownian motion**, is a process characterized by the following properties:

(i) B_H is a centered Gaussian process

(ii) B_H has stationary increments.

(iii) B_H is self-similar with index $H \in (0,1)$, that is for any $a > 0$,

$$\{B_H(at), \ t \geq 0\} \overset{\text{f.d.d.}}{=} \{a^H B_H(t), \ t \geq 0\}$$

These imply:

$$\mathbb{E}B_H(t)^2 = t^{2H}\mathbb{E}B(1)^2 = \sigma^2 t^{2H}$$

$$\mathbb{E}B_H(t_1)B_H(t_2) = \frac{\sigma^2}{2}[t_1^{2H} + t_2^{2H} - |t_1 - t_2|^{2H}]$$

Single Wiener-Itô stochastic integral

To represent Fractional Brownian motion we'll use a *Wiener-Itô stochastic integral* with respect to standard Brownian motion $B(t),\ t \geq 0$:

$$I = \int_{\mathbb{R}} f(x) dB(x)$$

This integral is defined first for each simple *non-random* $f \in L^2(\mathbb{R})$ and then defined by isometry as a limit in $L^2(\Omega)$.

Since I is a limit of Gaussian variables, I is also Gaussian, with mean 0 and variance $\|f\|_2^2 = \int_{\mathbb{R}} f(x)^2 dx$.

Representation of Fractional Brownian motion

Fractional Brownian motion with $1/2 < H < 1$ can be represented as a Wiener-Itô integral with respect to standard Brownian motion,

$$\begin{aligned} B_H(t) &= C_H \int_{\mathbb{R}} \left((t-x)_+^{H-1/2} - (-x)_+^{H-1/2} \right) dB(x) \\ &= C'_H \int_{\mathbb{R}} \left(\int_0^t (s-x)_+^{H-3/2} ds \right) dB(x) \end{aligned}$$

Using this representation, it is straightforward to check this process is fractional Brownian motion.

It is Gaussian, has stationary increments, is self-similar with Hurst parameter H.

More convenient parametrization

Fractional Brownian motion with $1/2 < H < 1$ will now be parametrized buy γ instead of H as follows:
Instead of

$$B_H(t) = C'_H \int_{\mathbb{R}} \left(\int_0^t (s-x)_+^{H-3/2} ds \right) dB(x)$$

we write:

$$B_H(t) = C'_H \int_{\mathbb{R}} \left(\int_0^t (s-x)_+^{\gamma} ds \right) dB(x)$$

where

$$\gamma = H - 3/2.$$

Here:

$$1/2 < H < 1 \Leftrightarrow -1 < \gamma < -1/2.$$

- Brownian motion
- Fractional Brownian motion

are Gaussian processes.

The non-Gaussian processes

- The Rosenblatt process
- The generalized Rosenblatt process

will be constructed using a *double Wiener-Itô* integral.

Double Wiener-Itô integral

Thus, to construct a non-Gaussian self-similar process with stationary increments, we will use a *double Wiener-Itô* integral:

$$I = \int_{\mathbb{R}^2}' f(x,y)dB(x)dB(y)$$

where the prime means to exclude the diagonal $\{x = y\}$, and f satisfies $\iint_{\mathbb{R}^2} f(x,y)^2 dxdy < \infty$.

The integral I is defined again as a limit of sums

$$I = \lim \sum_{n,m}{}' f(x_n, x_m)(B(x_n) - B(x_{n-1}))(B(x_m) - B(x_{m-1}))$$

where $\{x_n\}$ forms a partition of $[a, b]$ and where we exclude diagonals.

Representation of the Rosenblatt process

The Rosenblatt process, $\{Z_{\gamma,\gamma}(t),\ t \geq 0\}$ is a NON-Gaussian self-similar process with stationary increments, which is given by the *double* Wiener-Itô integral

$$Z_{\gamma,\gamma}(t) = A_\gamma \int'_{\mathbb{R}^2} \left(\int_0^t (s-x_1)_+^\gamma (s-x_2)_+^\gamma ds \right) dB(x_1) dB(x_2)$$

It is again "straightforward" to check that $Z_{\gamma,\gamma}$ has stationary increments and is self-similar with parameter

$$H = 2\gamma + 1 + 1/2 + 1/2 = 2\gamma + 2,$$

so that

$$1/2 < H < 1 \Leftrightarrow -3/4 < \gamma < -1/2.$$

Note that $\gamma < 0$.

The Rosenblatt distribution

We first focus on the marginal distribution of $Z_{\gamma,\gamma}$ at time $t = 1$, i.e. the distribution of the random variable

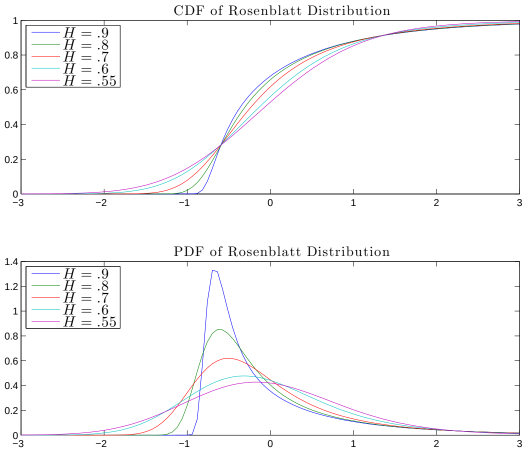
$$Z = Z_{\gamma,\gamma}(1) = Z_H(1)$$

called the *Rosenblatt distribution*. As defined, this distribution is standardized.

$$\mathbb{E}Z = 0$$
$$\mathbb{E}Z^2 = 1$$

However, Z is not Gaussian, and its distribution is not known in closed form.

Plots of the CDF and PDF of the Rosenblatt distribution



These are tabulated as well! (Highest curve $H = 0.9$)

A mystery !!!

It seems that the CDFs of the Rosenblatt distributions intersect at the same point **for any** H. In fact, this is true even for extremes:

For $H = 1$: $Z_1 = \text{standardized chi squared}$

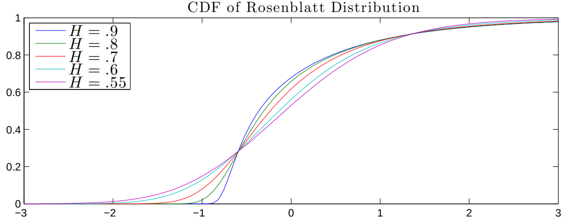
For $H = 1/2$: $Z_{1/2} = N(0,1)$:

$$P(Z_1 \le -0.6256) = P(Z_{1/2} \le -0.6256) = 0.2658.$$

Conjecture:

$$\forall \;\; 1/2 < H < 1 : \quad P(Z_H \le -0.6256) = 0.2658.$$

There seems to be a second common point: $P(Z_H \le 1.3552) = 0.9123$.



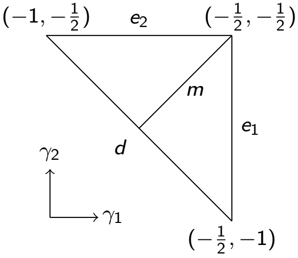
Generalized Rosenblatt process

Consider the *standardized* generalized Rosenblatt process

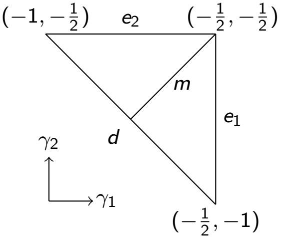
$$Z_{\gamma_1,\gamma_2}(t) = A(\gamma_1,\gamma_2)\int_{\mathbb{R}^2}' \int_0^t (s-x_1)_+^{\gamma_1}(s-x_2)_+^{\gamma_2} ds\ dB(x_1)dB(x_2),$$

$A(\gamma_1,\gamma_2)$ is chosen so that $\mathrm{Var}[Z_{\gamma_1,\gamma_2}(1)] = 1$, ($\mathbb{E}Z_{\gamma_1,\gamma_2}(t) = 0$ automatically), where

$$(\gamma_1,\gamma_2) \in \boldsymbol{\Delta} = \{(\gamma_1,\gamma_2), -1 < \gamma_1,\gamma_2 < -1/2,\ \gamma_1+\gamma_2 > -3/2\}$$



Basic Properties



Basic properties:

- When $\gamma_1 = \gamma_2 = \gamma$, then $Z_{\gamma,\gamma}(t)$ is the Rosenblatt process.
- $Z_{\gamma_1,\gamma_2}(t)$ is H-self-similar with stationary increments, with

$$H = \gamma_1 + \gamma_2 + 2 \in (1/2, 1).$$

- H is close to 1 when γ_1 and γ_2 are both close to $-1/2$.
- Symmetry $Z_{\gamma_1,\gamma_2}(t) = Z_{\gamma_2,\gamma_1}(t)$.

Do Generalized Rosenblatt Processes Form a Richer Class?

$$Z_{\gamma_1,\gamma_2}(t) = A(\gamma_1,\gamma_2) \int_{\mathbb{R}^k}' \int_0^t (s-x_1)_+^{\gamma_1}(s-x_2)_+^{\gamma_2} ds\ dB(x_1)dB(x_2).$$

Recall $\mathrm{Var}[Z_{\gamma_1,\gamma_2}(1)] = 1$.

Goal: show

$$\{Z_{\gamma,\gamma}\} \subsetneq \{Z_{\gamma_1,\gamma_2}\}.$$

Fixing $H = 2\gamma + 2 = \gamma_1 + \gamma_2 + 2$, the covariance structures are the same:

$$\mathrm{Cov}[Z_{\gamma,\gamma}(s), Z_{\gamma,\gamma}(t)] = \mathrm{Cov}[Z_{\gamma_1,\gamma_2}(s), Z_{\gamma_1,\gamma_2}(t)] = \frac{1}{2}\left(|s|^{2H} + |t|^{2H} - |s-t|^{2H}\right).$$

Idea: compute the third cumulant (moment) $\mu_3(\gamma_1,\gamma_2) = \mathbb{E}Z_{\gamma_1,\gamma_2}(1)^3$.

Show $\mu_3(\gamma_1,\gamma_2)$ varies on the line $\gamma_1 + \gamma_2 + 2 = H$ (H fixed).

$$k\text{-th cumulant of } \int_{\mathbb{R}^2}' f(x_1, x_2) dB(x_1) dB(x_2)$$
$$= 2^{k-1}(k-1)! \int_{\mathbb{R}^k} \widetilde{f}(x_1, x_2)\widetilde{f}(x_2, x_3) \dots \widetilde{f}(x_{k-1}, x_k)\widetilde{f}(x_k, x_1) dx_1 \dots dx_k.$$

where

$$\widetilde{f}(x_1, x_2) = \frac{1}{2}\left(f(x_1, x_2) + f(x_2, x_1)\right)$$

is the symmetrized version of $f(x_1, x_2)$.

Third cumulant of $Z_{\gamma_1,\gamma_2}(1)$ as a function of (γ_1, γ_2)

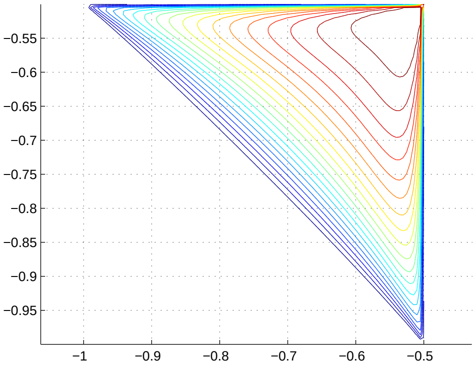


Figure : $\mu_3(\gamma_1, \gamma_2)$ contour on $\mathbf{\Delta} = \{(\gamma_1, \gamma_2) : -1 < \gamma_1, \gamma_2 < -1/2, \gamma_1 + \gamma_2 > -3/2\}$. Values in red are higher than values in blue.

Interesting finding: $\mu_3(\gamma_1, \gamma_2) \approx 0$ when (γ_1, γ_2) is close to the boundaries (except the northeast corner).

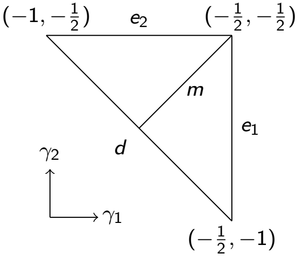
Question

How does the generalized Rosenblatt process

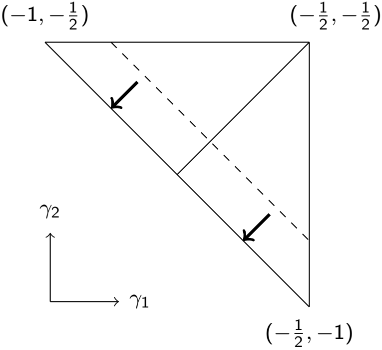
$$Z_{\gamma_1,\gamma_2}(t) = A(\gamma_1,\gamma_2) \int'_{\mathbb{R}^k} \int_0^t (s-x_1)_+^{\gamma_1}(s-x_2)_+^{\gamma_2} ds \; dB(x_1)dB(x_2)$$

with $\mathrm{Var}[Z_{\gamma_1,\gamma_2}(1)] = 1$, behave (in distribution) as (γ_1,γ_2) approaches the boundary of $\mathbf{\Delta}$?

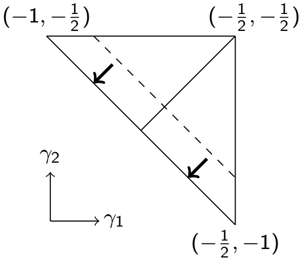
$$\mathbf{\Delta} = \{(\gamma_1,\gamma_2), -1 < \gamma_1, \gamma_2 < -1/2, \; \gamma_1+\gamma_2 > -3/2\}$$



Diagonal Boundary



Diagonal Boundary



Theorem. (Bai Taqqu 2016)

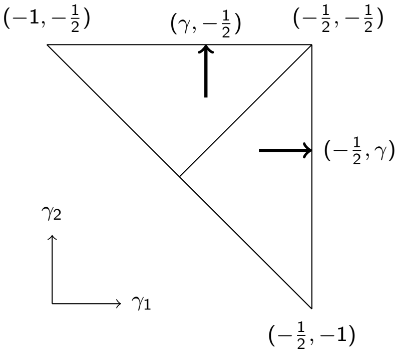
When $\gamma_1 + \gamma_2 \to -3/2$ *(*$H \to 1/2$*) with* $\gamma_1, \gamma_2 > -1 + \epsilon$ *for arbitrarily fixed small* $\epsilon > 0$*, we have*

$$Z_{\gamma_1,\gamma_2}(t) \Rightarrow B(t),$$

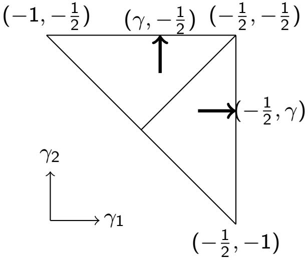
$\Rightarrow$*: weak convergence in* $C[0,1]$*,* $B(t)$*: a standard Brownian motion.*

Explanation: as $H \downarrow 1/2$, LRD $\downarrow$ SRD. Brownian motion is the typical SRD limit.

Side Boundary



Side Boundary



Theorem. (Bai Taqqu 2016)

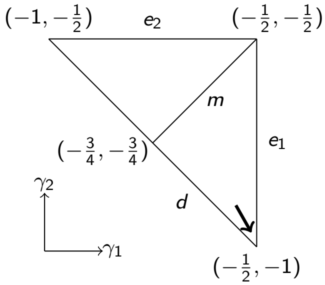
When $(\gamma_1,\gamma_2)\to(-1/2,\gamma)$ *or* $(\gamma_1,\gamma_2)\to(\gamma,-1/2)$, *where* $-1<\gamma<-1/2$ *, we have*

$$Z_{\gamma_1,\gamma_2}(t)\Rightarrow WB_{\gamma+3/2}(t),$$

$B_{\gamma+3/2}(t)$*: standard fractional Brownian motion with Hurst exponent* $\gamma+3/2$*;*
W*: a standard normal random variable;* $B_{\gamma+3/2}(t)$ *and* W *are independent.*

Explanation: when $\gamma_1\uparrow-1/2$, the moving average w.r.t. $dB(x_1)$ gets so extremely LRD that it becomes frozen. $Z_{\gamma_1,\gamma_2}(t)=A(\gamma_1,\gamma_2)\int'_{\mathbb{R}^k}\int_0^t(s-x_1)_+^{\gamma_1}(s-x_2)_+^{\gamma_2}ds\, dB(x_1)dB(x_2)$.

Corner 1

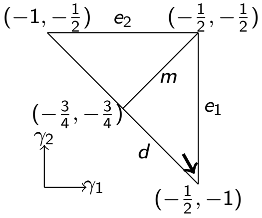


The limit depends on how one gets to the corner!
Let $(\gamma_1, \gamma_2) \to (-1/2, -1)$ in such a way that

$$\frac{\gamma_1 + \gamma_2 + 3/2}{\gamma_2 + 1} = 1 + \frac{\gamma_1 + 1/2}{\gamma_2 + 1} \to \rho \in [0, 1],$$

When $\rho = 0$, the line coincides with the diagonal edge d of the triangle $\mathbf{\Delta}$, which has slope -1. When $\rho = 1$, the line coincides with the vertical side e_1 of $\mathbf{\Delta}$, which has slope $-\infty$.

Corner 1



Theorem. (Bai Taqqu 2016)
If $(\gamma_1, \gamma_2) \to (-1/2, -1)$ *in such a way that*

$$\frac{\gamma_1 + \gamma_2 + 3/2}{\gamma_2 + 1} = 1 + \frac{\gamma_1 + 1/2}{\gamma_2 + 1} \to \rho \in [0, 1],$$

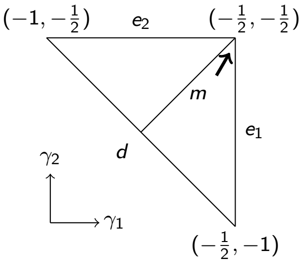
then

$$Z_{\gamma_1,\gamma_2}(t) \Rightarrow X_\rho(t) := \rho^{1/2} W B(t) + (1-\rho)^{1/2} B'(t)$$

W: a standard normal random variable, $B(t)$ and $B'(t)$: standard Brownian motions
W, $B(t)$ and $B'(t)$ are independent.

The limit $X_\rho(t)$ is an independent linear combination of the two limits obtained earlier.

Corner 2

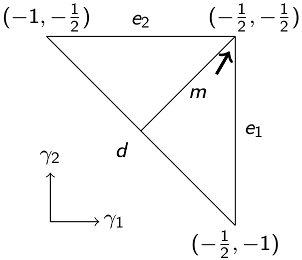


Suppose that $\gamma_1 \geq \gamma_2$. Let $(\gamma_1, \gamma_2) \to (-1/2, -1/2)$ in such a way that

$$\frac{\gamma_1 + 1/2}{\gamma_2 + 1/2} \to \rho \in [0, 1].$$

When $\rho = 0$, the line coincides with the vertical side e_1 of $\mathbf{\Delta}$, which has slope $+\infty$. When $\rho = 1$, the line coincides with the middle line m, which has slope 1.

Corner 2



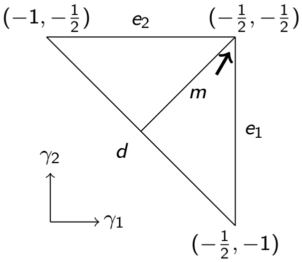
Theorem. (Bai Taqqu 2016)
Suppose that $\gamma_1 \geq \gamma_2$. *If* $(\gamma_1, \gamma_2) \to (-1/2, -1/2)$ *in such a way that*

$$\frac{\gamma_1 + 1/2}{\gamma_2 + 1/2} \to \rho \in [0, 1], \qquad \textit{then}$$

$$Z_{\gamma_1,\gamma_2}(t) \Rightarrow Y_\rho(t) := t \cdot \left[\frac{(\rho+1)^{-1} + (2\sqrt{\rho})^{-1}}{\sqrt{2(\rho+1)^{-2} + (2\rho)^{-1}}} \cdot X_1 + \frac{(\rho+1)^{-1} - (2\sqrt{\rho})^{-1}}{\sqrt{2(\rho+1)^{-2} + (2\rho)^{-1}}} \cdot X_2 \right],$$

X_1 *and* X_2*: independent standardized* χ_1^2 *random variables.*

Corner 2



• When $\rho = 0$, the line coincides with the vertical side e_1 of $\mathbf{\Delta}$, which has slope $+\infty$. In this case, the limit $Y_\rho(t) = \frac{t}{\sqrt{2}}(X_1 - X_2)$, X_1, X_2 i.i.d. chi squared, which has the same distribution as $t\,(WB)$, where W and B are two independent standard normal random variables

• When $\rho = 1$, the line coincides with the middle line m, which has slope 1. In this case, the limit $Y_\rho(t)$ reduces to tX_1, where X_1 is a standardized chi-squared random variable with one degree of freedom.

No convergence in $L^2(\Omega)$

Theorem.
In the previous theorems, the weak convergence cannot be extended to convergence in $L^2(\Omega)$, nor even to convergence in probability.

Sketch of the proof:

• Convergence in $L^2(\Omega)$ implies convergence in probability. But on a fixed-order Wiener chaos, they are equivalent:
Convergence in probability $\Rightarrow$ Tightness (+ Hypercontractivity on fixed-order Chaos) $\Rightarrow$ Boundedness in $L^p(\Omega)$, $\forall p > 2$, $\Rightarrow$ Uniform integrability $\Rightarrow$ Convergence in $L^2(\Omega)$ (Schreiber (1969) or Nourdin Rosinski (2014), Lemma 2.1).

• No Cauchy convergence in $L^2(\Omega)$ as

$$\limsup_{(\alpha_1,\alpha_2),(\gamma_1,\gamma_2)\to \text{ boundary point}} \mathbb{E}\left(Z_{\alpha_1,\alpha_2}(1) - Z_{\gamma_1,\gamma_2}(1)\right)^2 > 0.$$

Rates of convergence

• Let $d_{TV}(X, Y)$ denote the *total variation distance* between the distributions of random variables X and Y, namely

$$d_{TV}(X, Y) = \sup_{S \in \mathcal{B}(\mathbb{R})} |P(X \in S) - P(Y \in S)|,$$

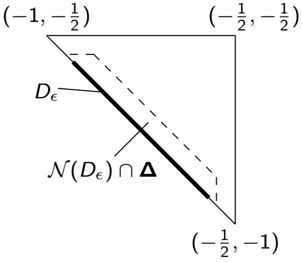
where $\mathcal{B}(\mathbb{R})$ denotes the Borel sets on $\mathbb{R}$.

• Let $d_W(X, Y)$ denote the *Wasserstein distance* between the distributions of random variables X and Y, namely,

$$d_W(X, Y) = \sup_{h \in \mathcal{L}} \{|\mathbb{E}h(X) - \mathbb{E}h(Y)|\},$$

where $\mathcal{L}$ is the class of 1-Lipschitz functions ($h \in \mathcal{L}$ if $|h(x) - h(y)| \leq |x - y|$).

Diagonal convergence

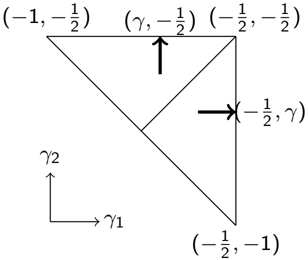


Theorem.
Let $Z_{\gamma_1,\gamma_2} = Z_{\gamma_1,\gamma_2}(1)$, and let N be a standard normal random variable. Let (γ_1, γ_2) approach the line segment $D_\epsilon := \{\gamma_1 + \gamma_2 + 3/2 = 0 : \gamma_1, \gamma_2 > -1 + \epsilon\}$. Then there exists a neighborhood $\mathcal{N}(D_\epsilon)$ of D_ϵ, such that when $(\gamma_1, \gamma_2) \in \mathcal{N}(D_\epsilon) \cap \boldsymbol{\Delta}$, we have

$$C_1(\gamma_1 + \gamma_2 + 3/2)^{3/2} \le d_{TV}(Z_{\gamma_1,\gamma_2}, N) \le C_2(\gamma_1 + \gamma_2 + 3/2)^{3/2}.$$

Nourdin Peccati (2015): $d_{TV}(Z_{\gamma_1,\gamma_2}, N) \asymp \max\left(|\mathbb{E}Z^3_{\gamma_1,\gamma_2}|, |\mathbb{E}Z^4_{\gamma_1,\gamma_2} - 3|\right)$.
There are partial results on other boundaries.

Side convergence



Let Y be the limit (product of independent standard normal) as

$$(\gamma_1, \gamma_2) \to (-1/2, \gamma), \quad -1 < \gamma < -1/2$$

(away from the corners). Then

$$d_W(Z_{\gamma_1,\gamma_2}(1), Y) = O\left(\left(-\gamma_1 - 1/2\right)^{1/2}\right).$$

We use Eichelsbacher and Thäle (2014):

$$d_W(Z_{\gamma_1,\gamma_2}(1), Y) \le C\left(1 + \frac{1}{6}\kappa_3(Z_{\gamma_1,\gamma_2})^2 - \frac{1}{3}\kappa_4(Z_{\gamma_1,\gamma_2}) + \frac{1}{120}\kappa_6(Z_{\gamma_1,\gamma_2})\right)^{1/2}.$$

Recently Arras, Azmoodeh, Poly and Swan (2016) obtained the rate of convergence when the limit is $\sum_{i=1}^{q} \alpha_i X_i$ where X_i's are standardized chi-square random variables with one degree of freedom. Appying their Theorem 3.1 to the convergence of $(\gamma_1, \gamma_2) \in \mathbf{\Delta}$ to the corner $(-1/2, -1/2)$, they obtained as $(\gamma_1, \gamma_2) \to (-1/2, -1/2)$ that

$$d_W(Z_{\gamma_1,\gamma_2}(1), Y_\rho(1)) = O((-\gamma_1 - 1/2)^{1/2}),$$

where $Y_\rho(1)$ is the limit (recall $\rho \in [0, 1]$ parameterizes the direction). See Example 3.2 of Arras et al. (2016).

Thank you for your attention!

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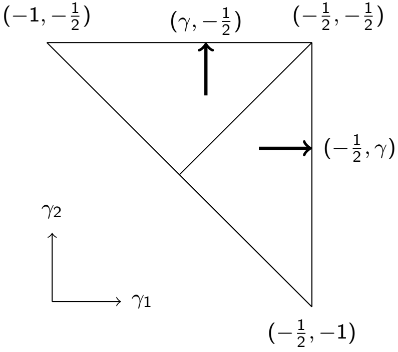
Alternative proof for side convergence



The method-of-moments proof gives little intuitive insight of the convergence. We also give an alternate proof of the convergence on the diagonal side boundary. The proof is based on discretization which removes the singularities at $s = x_1$ and $s = x_2$ of the integrand, so that one is able to interchange the integration orders between $\int'_{\mathbb{R}^2} \cdot B(dx_1)B(dx_2)$ and $\int_0^t \cdot ds$. Then one uses the triangular approximation described at the end of the proof.

The Trivial process

Any right-continuous self-similar process $X(t)$ with $\mathbb{E}|X(t)|^2 < \infty$, stationary increments and $H = 1$ is the trivial process $X(t) = tX(1)$ a.s..

Indeed, due to stationary increments

$$\mathbb{E}X(t)X(s) = \frac{1}{2}\left[\mathbb{E}X(t)^2 + \mathbb{E}X(s)^2 - \mathbb{E}X(t-s)^2\right]$$

Using this and self-similarity

$$\begin{aligned}&\mathbb{E}[X(t) - tX(1)]^2 = \mathbb{E}X(t)^2 - 2t\mathbb{E}X(1)X(t) + t^2\mathbb{E}X(1)^2\\=&t^2\mathbb{E}X(1)^2 - 2t\frac{1}{2}\left[\mathbb{E}X(1)^2 + t^2\mathbb{E}X(1)^2 - (t-1)^2\mathbb{E}X(1)^2\right] + t^2\mathbb{E}X(1)^2\\=&t^2\mathbb{E}X(1)^2 - 2t^2\mathbb{E}X(1)^2 + t^2\mathbb{E}X(1)^2 = 0\end{aligned}$$

We get for any fixed $t > 0$,

$$X(t) = tX(1) \quad a.s.,$$

and then $X(t) = tX(1)$ a.s. for all t by right-continuity.

Tool: Cumulant Formula

The m-th cumulant ($m \geq 2$) of $\sum_{i=1}^{n} c_i Z_{\gamma_1,\gamma_2}(t_i)$, $c_i \in \mathbb{R}$, $t_i \in [0,\infty)$, equals

$$\kappa_m\left(\sum_{i=1}^{n} c_i Z_{\gamma_1,\gamma_2}(t_i)\right) = \frac{1}{2}(m-1)!A^m C_m(\gamma_1,\gamma_2;\mathbf{t},\mathbf{c}),$$

where

$$A(\gamma_1,\gamma_2) = \Big[(\gamma_1+\gamma_2+2)(2(\gamma_1+\gamma_2)+3)\Big]^{1/2}$$
$$\times\Big[\mathrm{B}(\gamma_1+1,-\gamma_1-\gamma_2-1)\mathrm{B}(\gamma_2+1,-\gamma_1-\gamma_2-1)+\mathrm{B}(\gamma_1+1,-2\gamma_1-1)\mathrm{B}(\gamma_2+1,-2\gamma_2-1)\Big]^{-1/2}.$$

$$C_m(\gamma_1,\gamma_2;\mathbf{t},\mathbf{c}) = \sum_{\sigma\in\{1,2\}^m}\sum_{i_1,\ldots,i_m=1}^{n} c_{i_1}\ldots c_{i_m}\int_0^{t_{i_1}} ds_1\ldots\int_0^{t_{i_m}} ds_m$$
$$\prod_{j=1}^{m}\Big[(s_j-s_{j-1})_+^{\gamma_{\sigma_j}+\gamma_{\sigma'_{j-1}}+1}\mathrm{B}(\gamma_{\sigma'_{j-1}}+1,-\gamma_{\sigma_j}-\gamma_{\sigma'_{j-1}}-1)$$
$$+(s_{j-1}-s_j)_+^{\gamma_{\sigma_j}+\gamma_{\sigma'_{j-1}}+1}\mathrm{B}(\gamma_{\sigma_j}+1,-\gamma_{\sigma_j}-\gamma_{\sigma'_{j-1}}-1)\Big],$$

where

$$\mathrm{B}(x,y) = \int_0^1 u^{x-1}(1-u)^{y-1}du, \quad x,y>0,$$

is the beta function, the sum $\sum_{\sigma\in\{1,2\}^m}$ runs over $\sigma=(\sigma_1,\ldots,\sigma_m)$ with $\sigma_i = 1$ or 2, and σ' is the complement of σ, namely, $\sigma'_i = 1$ if $\sigma_i = 2$ and $\sigma'_i = 2$ if $\sigma_i = 1$, $i = 1,\ldots,m$. Moreover $\sigma'_0 = \sigma'_m$ and $s_0 = s_m$, $i = 1,\ldots,m$.

No limit at corners if not following straight-line directions

To get limits for $Z_{\gamma_1,\gamma_2}(t)$ at the corners, the parameters (γ_1, γ_2) have to approach the corners along some straight-line direction. If not, the limit will not exist.

Proof.

Suppose that the direction parameter $\rho(\gamma_1, \gamma_2)$ does not converge as (γ_1, γ_2) approaches the corner $(-\frac{1}{2}, -1)$. Then there are two subsequences of (γ_1, γ_2), such that $\rho(\gamma_1, \gamma_2)$ of the first subsequence converges to ρ_1 and $\rho(\gamma_1, \gamma_2)$ of the second subsequence converges to ρ_2, with $\rho_1 \neq \rho_2$. By the theorem, the corresponding processes $Z_{\gamma_1,\gamma_2}(t)$ converge to two different limits. Therefore, the original process $Z_{\gamma_1,\gamma_2}(t)$ does not converge if (γ_1, γ_2) does not follow a straight-line direction. □