Some connections between max-stable processes, random sets and risk measures

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joint work with Ilya Molchanov[2]

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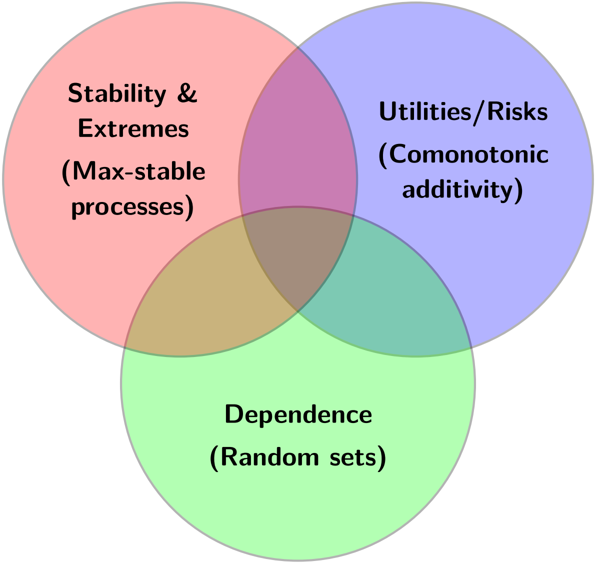
Workshop on Dependence, Stability and Extremes
Fields Institute, Toronto – May 4, 2016


Stability & Extremes
(Max-stable processes)
Utilities/Risks
(Comonotonic additivity)
Dependence
(Random sets)

Background on random sets

Random sets

- **Notation.** T lcscH space (e.g. $\mathbb{R}^d$, $[0,\infty)$, finite, ...),

  $\mathcal{F}$ = set of closed subsets,
  $\mathcal{K}$ = set of compact subsets,
  $\mathcal{G}$ = set of open subsets

- random closed set = random element $\Xi \in \mathcal{F}$
  (with σ-alg. from Fell-topology)

- its law is uniquely specified by the hitting probabilities

$$\theta_{\Xi}(K) = \mathbf{P}\{\Xi \cap K \neq \emptyset\}, \quad K \in \mathcal{K}$$

  (capacity functional of Ξ)

Choquet theorem

- **Choquet theorem.** $\theta : \mathcal{K} \to [0,1]$ is the capacity functional of a random closed set if and only if
  - $\theta(\emptyset) = 0$
  - θ is usc: $\theta(K_n) \downarrow \theta(K)$ for $K_n \downarrow K$
  - θ is completely alternating (with respect to $\cup$)

- **Complete alternation.** $=$ n-alternation for any $n \geqslant 1$

1-alternating: $\Delta_K \theta(L) = \theta(L) - \theta(K \cup L) \leqslant 0$

2-alternating: $\Delta_{K_1} \Delta_{K_2} \theta(L)$
$= \theta(L) - \theta(L \cup K_1) - \theta(L \cup K_2) + \theta(L \cup K_1 \cup K_2) \leqslant 0$

⋮

n-alternating: $\Delta_{K_1} \Delta_{K_2} \ldots \Delta_{K_n} \theta(L) \leqslant 0$

Setup for this talk
(+ further background)

Max-stable processes

- $\{\xi(t)\}_{t \in T}$ random element in a space of paths on lcscH space T (e.g. $\mathbb{R}^d$, $[0, \infty)$, finite set)

- max-stability (std. case):

$$\xi_1 \vee \xi_2 \vee \ldots \vee \xi_n \overset{\mathcal{D}}{=} n\,\xi \quad \text{for} \quad \xi_i \overset{\text{i.i.d.}}{\sim} \xi$$

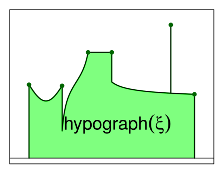
- convenient setup? upper semicontinuous paths

  [Vervaat '88, Norberg '86, Salinetti/Wets '86, O'Brien/Vervaat '91, Resnick/Roy '91, Molchanov '05, . . . , Lacaux/Samorodnitsky '16+, Sabourin/Segers '16, ...]

Topology for usc trajectories

Two equivalent concepts:

Upper semicts. paths $\xi \geqslant 0$
with Fell-top. on hypographs



Sup-measures $X \geqslant 0$
with sup-vague toplogy

$$\limsup X_n(K) \leqslant X(K)$$
$$\liminf X_n(G) \geqslant X(G)$$

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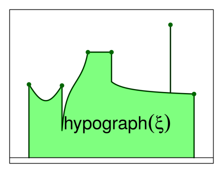

sup-integral

$$X(K) = \bigvee_{t \in K} \xi(t)$$

Sup-measures $X \geqslant 0$
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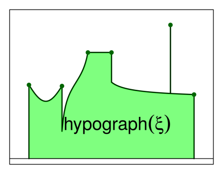

sup-integral

$$X(K) = \bigvee_{t \in K} \xi(t)$$

sup-derivative

$$\xi(t) = X(\{t\})$$

Sup-measures $X \geqslant 0$
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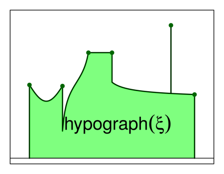

sup-integral

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→

Sup-measures $X \geqslant 0$ with sup-vague toplogy

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$$\liminf X_n(G) \geqslant X(G)$$

←

sup-derivative

$$\xi(t) = X(\{t\})$$

Central object. X = a locally finite max-stable random sup-measure

Example

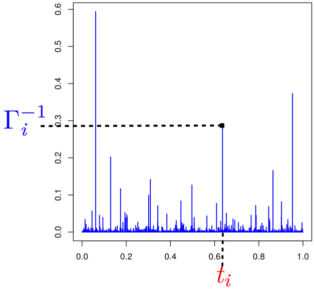
A max-stable rsm on $T = [0, 1]$:

$$\xi(t) = \bigvee_{i \geqslant 1} \Gamma_i^{-1} \mathbb{1}_{\{t_i\}}(t),$$

$$X(K) = \bigvee_{i \geqslant 1} \Gamma_i^{-1} \mathbb{1}_{\{t_i\} \cap K \neq \emptyset},$$

with independent ingredients

- $\{\Gamma_i, i \geqslant 1\} \sim \mathrm{PPP}(\text{Leb.})$ on $\mathbb{R}_+$
- $\{t_i, i \geqslant 1\} \sim$ i.i.d. $\text{Unif}([0, 1])$



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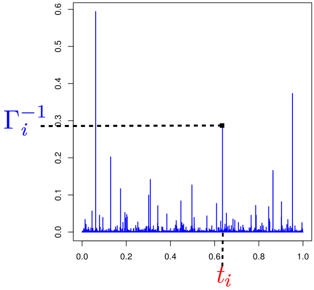


Special property of this example.

- completely random = independent peaks
  $X(K_i)$, $i = 1, \dots, m$ are indep. for disjoint K_i
- cf. [Stoev/Taqqu '06] / control measure $\mu = \text{Unif}([0,1])$

Extremal coefficient functional

$f = \mathbb{1}_K$, $K \in \mathcal{K}$ (compact)

Extremal integral:
as in [Stoev/Taqqu '06]

$$\int^e \mathbb{1}_K \, dX = X(K) = \bigvee_{t \in K} \xi(t)$$

Extremal coefficient functional:

$$\mathbf{P}\left\{X(K) \leqslant x\right\} = \exp\left(-\frac{\theta(K)}{x}\right)$$

Example. If X is completely random with control measure μ, then

$$\theta(K) = \mu(K).$$

Question. Does θ determine the distribution of X?

Tail dependence functional

$f \in \mathrm{USC}_0(T)$ (usc, bounded, compact support)

Extremal integral: $$\int^e f \, dX = \sup_{x>0} x \, X(\{f \geqslant x\}) = \bigvee_{t \in T} f(t)\xi(t)$$

[Gerritse '96]

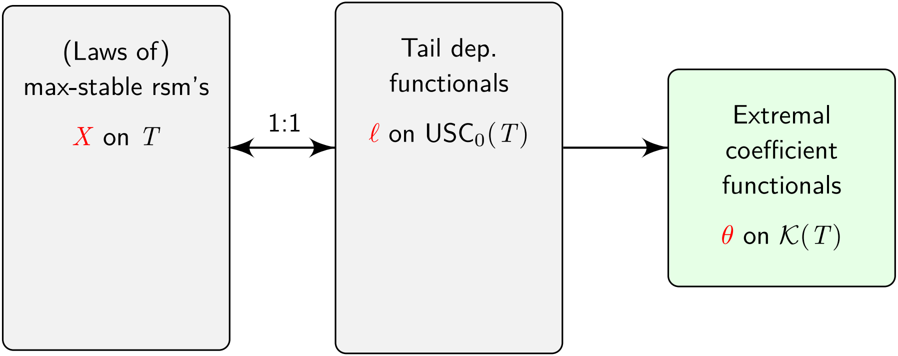
Tail dependence functional: $$\mathbf{P}\left\{\int^e f \, dX \leqslant x\right\} = \exp\left(-\frac{\ell(f)}{x}\right)$$

Example. If X is completely random with control measure μ, then

$$\ell(f) = \int f d\mu.$$

Answer. The tdf ℓ determines the distribution of X.

Plan for the (next part of this) talk



$$\mathbf{P}\left\{\int^e f \; dX \leqslant x\right\} = \exp\left(-\frac{\ell(f)}{x}\right) \qquad \mathbf{P}\left\{X(K) \leqslant x\right\} = \exp\left(-\frac{\theta(K)}{x}\right)$$

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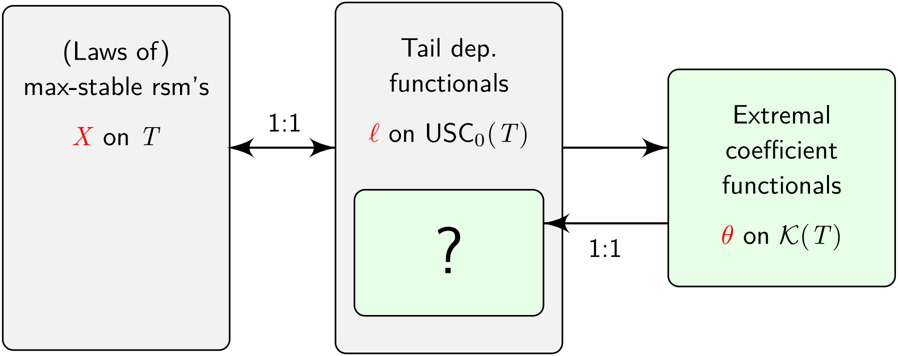

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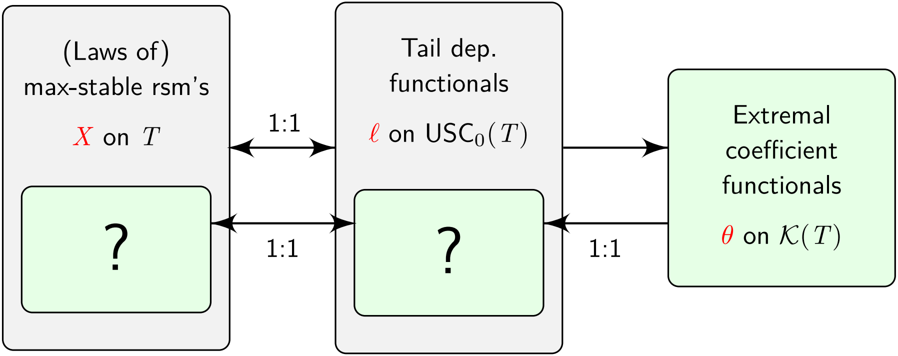

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Properties of ℓ and θ

Tail dependence functional

ℓ on USC_0

$$\mathbf{P}\left\{\int^e f \; dX \leqslant x\right\} = \exp\left(-\frac{\ell(f)}{x}\right)$$

- homogeneous: $\ell(cf) = c\ell(f)$
- usc: $f_n \downarrow f \Rightarrow \ell(f_n) \downarrow \ell(f)$
- max-completely alternating

Extremal coefficient functional

θ on $\mathcal{K}$

$$\mathbf{P}\left\{X(K) \leqslant x\right\} = \exp\left(-\frac{\theta(K)}{x}\right)$$

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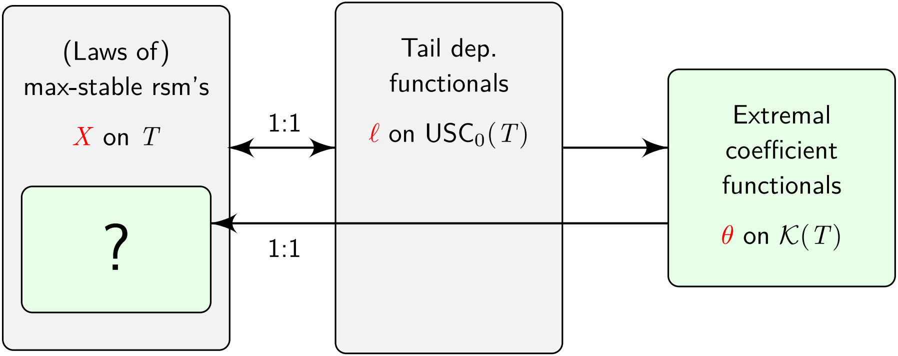
Complete alternation (Semigroup property)

ψ: $(S, \vee) \to \mathbb{R}$ satisfies for all $n \geqslant 1$, $s, s_1, \ldots, s_n \in S$

$$\left(\Delta_{s_1}\Delta_{s_2}\ldots\Delta_{s_n}\psi\right)(s) = \sum_{I \subset \{1,\ldots,n\}} (-1)^{|I|}\, \psi\left(s \vee \bigvee_{i \in I} s_i\right) \leqslant 0.$$

Plan



$$\mathbf{P}\left\{\int^e f\ dX \leqslant x\right\} = \exp\left(-\frac{\ell(f)}{x}\right) \qquad \mathbf{P}\left\{X(K) \leqslant x\right\} = \exp\left(-\frac{\theta(K)}{x}\right)$$

Question. Given an ecf θ, how to associate a max-stable rsm X?
Answer. Use Choquet theorem (for not necessarily finite capacities).

Theorem [Molchanov, S. 16+]

If θ is an ecf, then there exists a locally finite measure ν_θ on $\mathcal{F}$ with

$$\nu_\theta(\{F : F \cap K \neq \emptyset\}) = \theta(K), \qquad K \in \mathcal{K}.$$

The max-series

$$X(K) = \bigvee_{i \geqslant 1} \lambda_i^{-1} \mathbb{1}_{F_i \cap K \neq \emptyset}, \qquad K \in \mathcal{K},$$

where $\{(\lambda_i, F_i) : i \geqslant 1\} \sim \mathrm{PPP}(Leb. \times \nu_\theta)$ on $\mathbb{R}_+ \times \mathcal{F}$ is a max-stable rsm with ecf θ.

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Name.
Choquet random sup-measures

LePage representation

Theorem [Molchanov, S. 16+]

(a) Each max-stable rsm X can be decomposed as max-series

$$X \overset{\mathcal{D}}{=} \bigvee_{i \geqslant 1} \Gamma_i^{-1} Y_i$$

with independent ingredients

- $\{\Gamma_i, i \geqslant 1\} \sim \mathrm{PPP}(\text{Leb.})$ on $\mathbb{R}_+$
- $\{Y_i, i \geqslant 1\} \sim$ i.i.d. random sup-measures

(b) X is a Choquet rsm. $\Leftrightarrow$ The law of Y is supported by scaled indicator sup-measures

$$\{c \mathbb{1}_{F \cap K \neq \emptyset} : c \in \mathbb{R}_+,\ F \in \mathcal{F}\}.$$

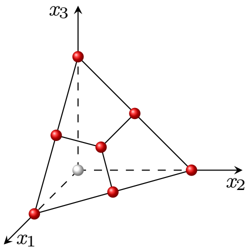
Corresponding functionals.
$\ell(f) = \mathbf{E}\left(\int^e f dY\right)$
$\theta(K) = \mathbf{E}\,(Y(K))$

Proof based on:
[Norberg '86, Davydov/Molchanov/Zuyev '08, deHaan '84]

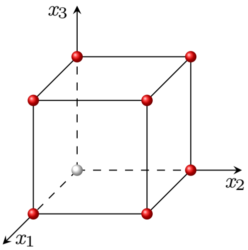
Special case $T = \{1, 2, \ldots, d\}$

$$\mathbf{P}\left\{\bigvee_{i=1}^{d} f(i) X(\{i\}) \leqslant 1\right\} = \exp\left(-\int_{S_+} \left(\bigvee_{i=1}^{d} f(i)\omega_i\right) H(d\omega)\right)$$

Support sets for the spectral measure H in the Choquet case:



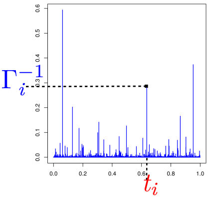
$S_+ = \{\omega \in \mathbb{R}_+^m : \|\omega\|_1 = 1\}$



$S_+ = \{\omega \in \mathbb{R}_+^m : \|\omega\|_\infty = 1\}$

“Spatial” examples of Choquet rsm’s

Example 1. Completely random max-stable rsm’s are Choquet rsm’s with ecf θ = control measure.

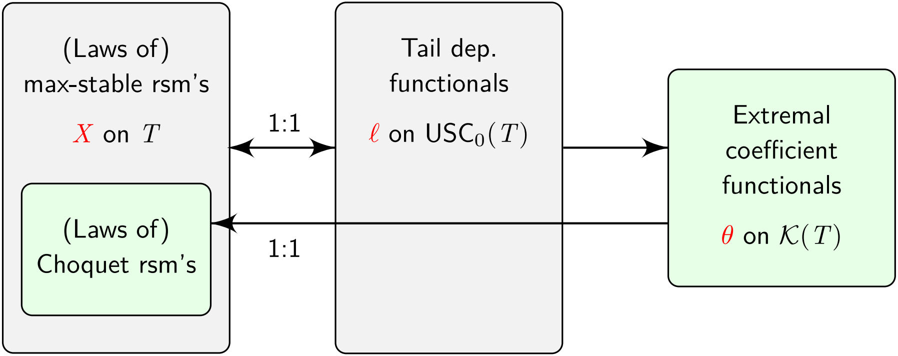


Example 2. [Lacaux/Samorodnitsky ’16+]:

Limit theorem. For a certain stationary SαS-process $X_1, X_2, \ldots$ (here $\alpha = 1$) with long-memory parameter β

$$\left(a_n^{-1} \bigvee_{j \in nK} X_j\right)_{K \in \mathcal{K}} \xrightarrow{\mathcal{D}} \begin{cases} \text{stationary, self-similar Choquet rsm with ecf} \\ \theta(K) = \int_0^\infty \mathbf{P}_\Xi\{\Xi + x \cap K \neq \emptyset\} \beta x^{\beta-1}\, dx, \\ \Xi = \text{image of a } (1-\beta)\text{-stable subordinator.} \end{cases}$$

Plan



$$\mathbf{P}\left\{\int^e f \, dX \leqslant x\right\} = \exp\left(-\frac{\ell(f)}{x}\right) \qquad \mathbf{P}\left\{X(K) \leqslant x\right\} = \exp\left(-\frac{\theta(K)}{x}\right)$$

Characterization of ℓ and θ

Tail dependence functional

ℓ on USC_0

$$\mathbf{P}\left\{\int^e f \; dX \leqslant x\right\} = \exp\left(-\frac{\ell(f)}{x}\right)$$

- homogeneous: $\ell(cf) = c\ell(f)$
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Extremal coefficient functional

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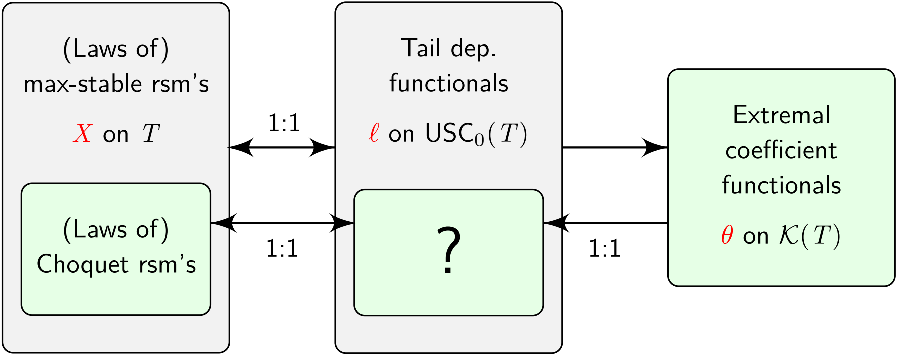
- $\theta(\emptyset) = 0$
- usc: $K_n \downarrow K \Rightarrow \theta(K_n) \downarrow \theta(K)$
- union-completely alternating

Theorem [Molchanov, S. 16+]

These properties are necessary and sufficient for ℓ (resp. θ) to be the tail dependence (resp. extremal coefficient functional) of a max-stable random sup-measure.

finite dim'l: [Ressel '11/'13, Molchanov '08]

Plan



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Question. Given an ecf θ, how to associate a tdf ℓ?
Answer. By the Choquet integral.

Theorem [Molchanov, S. 16+]

If θ is an ecf, then the Choquet integral

$$\ell(f) = \int f \, d\theta = \int_0^\infty \theta(\{f \geqslant x\}) \, dx, \qquad f \in \mathrm{USC}_0,$$

is a tdf with the property $\ell(\mathbb{1}_K) = \theta(K)$, $K \in \mathcal{K}$.

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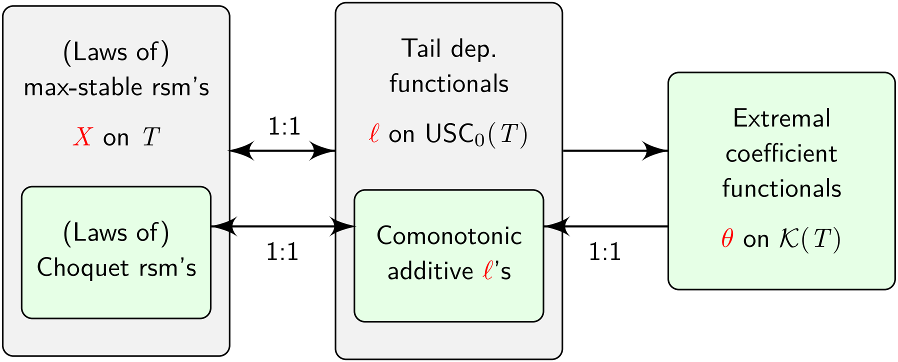
Important property. The Choquet integral is comonotonic additive:

$$\int f + g \, d\theta = \int f \, d\theta + \int g \, d\theta \qquad \text{for comonotonic } f \text{ and } g$$

Conversely, any comonotonic additive f'l is a Choquet integral. [Schmeidler '86]

Plan



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Association of Choquet rsm's to general max-stable rsm's

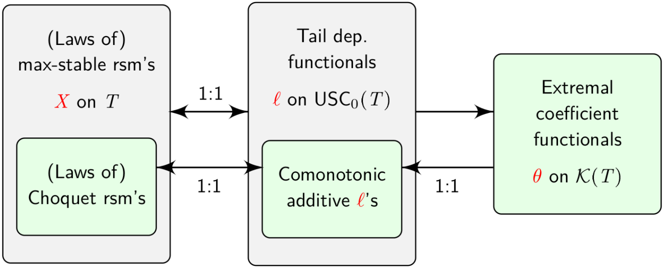
Theorem (Stochastic dominance property) [Molchanov, S. 16+]

Each max-stable rsm X can be coupled with a Choquet rsm $\widetilde{X}$, such that

$$\theta = \widetilde{\theta}, \text{ i.e. } \quad \mathbf{P}\{X(K) \leqslant t\} = \mathbf{P}\left\{\widetilde{X}(K) \leqslant t\right\}, \quad K \in \mathcal{K},\ t > 0$$

$$\ell \leqslant \widetilde{\ell}, \text{ i.e. } \quad \mathbf{P}\left\{\int^e f dX \leqslant t\right\} \geqslant \mathbf{P}\left\{\int^e f d\widetilde{X} \leqslant t\right\}, \quad f \in \mathrm{USC}_0,\ t > 0$$

Proof.



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$$\ell \leqslant \widetilde{\ell}, \text{ i.e. } \quad \mathbf{P}\{\textstyle\int^e f dX \leqslant t\} \geqslant \mathbf{P}\left\{\textstyle\int^e f d\widetilde{X} \leqslant t\right\}, \quad f \in \mathrm{USC}_0,\ t > 0$$

Proof. Dual representation of the tdf ℓ.

$$\ell(f) = \sup_{\mu \in \mathbf{M}} \int f d\mu \quad \text{with} \quad \mathbf{M} = \left\{\mu : \int f d\mu \leqslant \ell(f),\ f \in \mathrm{USC}_0\right\}$$

$$\widetilde{\ell}(f) = \sup_{\mu \in \widetilde{\mathbf{M}}} \int f d\mu \quad \text{with} \quad \widetilde{\mathbf{M}} = \left\{\mu : \mu(K) \leqslant \theta(K),\ K \in \mathcal{K}\right\}$$

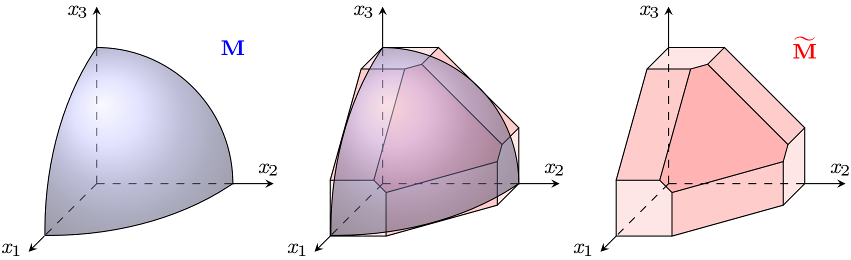
$$\mathbf{M} \subset \widetilde{\mathbf{M}}$$

□

Special case $T = \{1, 2, \ldots, d\}$

$$\mathbf{M} = \left\{ \mu \in \mathbb{R}_+^d : \langle f, \mu \rangle \leqslant \ell(f),\, f \in \mathbb{R}_+^d \right\}$$

$$\widetilde{\mathbf{M}} = \left\{ \mu \in \mathbb{R}_+^d : \langle \mathbb{1}_K, \mu \rangle \leqslant \ell(\mathbb{1}_K),\, K \subset \{1, \ldots, d\} \right\}$$



Ordered coupling for finite max-stable rsm's

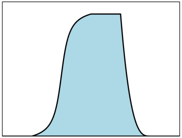
Proposition [Molchanov, S. 16+]

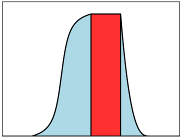
Let X be a finite max-stable rsm. Then there exist unique Choquet rsm's X_* and X^* (on the same probability space) such that

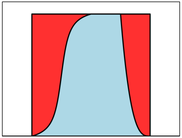
$$X_*(K) \leqslant X(K) \leqslant X^*(K) \quad \text{for all } K \in \mathcal{K},$$

and for any other Choquet rsm's X' and X'' with this property, $X'(K) \leqslant X_*(K)$ and $X^*(K) \leqslant X''(K)$ for all $K \in \mathcal{K}$.

Idea. (pathwise hypographs of Y, Y_*, Y^* in the LePage representation)







Set of optimal choices.

$$\text{argmax}(X) = \{t \in T : X(\{t\}) = X(T)\}$$

is a random closed set.

Corollary [Molchanov, S. 16+]

Let X be a finite max-stable rsm with LePage rep.

$$X \overset{\mathcal{D}}{=} \bigvee_{i \geqslant 1} \Gamma_i^{-1} Y_i.$$

Then

(a) $\text{argmax}(X)$ is independent of the maximal value $X(T)$.

(b) $\text{argmax}(X)$ and $\text{argmax}(Y)$ have the same distribution.

recovers and extends parts of [Resnick/Roy '91]

Summary

- **random sup-measures (rsm's)**
  - natural (?) framework for max-stability
  - links to random sets, utilities/risks
- **max-stable rsm's**
  - LePage representation
  - dual representation for tdf ℓ
- **Choquet rsm's**
  - tdf ℓ is comonotonic additive (Choquet integral wrt ecf θ)
  - Series/LePage representation (scaled indicator sup-measures)
  - association of Choquet rsm's
    $\Rightarrow$ stochastic dominance
    $\Rightarrow$ properties of argmax-set

Further aspects

I. Molchanov and K. Strokorb
Max-stable random sup-measures with comonotonic tail dependence
To appear in *Stoch. Proc. Appl.* Technical report URL http://arXiv.org/abs/1507.03476

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Thank you!

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LePage representation

Theorem

(a) *Each max-stable rsm X can be decomposed as max-series*

$$X \overset{\mathcal{D}}{=} \bigvee_{i \geqslant 1} \Gamma_i^{-1} Y_i$$

with independent ingredients

- $\{\Gamma_i, i \geqslant 1\} \sim \mathrm{PPP}(\textit{Leb.})$ *on* $\mathbb{R}_+$
- $\{Y_i, i \geqslant 1\} \sim$ *i.i.d. random sup-measures*

(b) X *is a Choquet rsm.* $\Leftrightarrow$ *The law of* Y *is supported by scaled indicator sup-measures*

$$\{c\mathbb{1}_{F \cap K \neq \emptyset} : c \in \mathbb{R}_+,\ F \in \mathcal{F}\}.$$

(c) X *is a Choquet rsm.* $\Leftrightarrow$ *For all* $f \in \mathrm{USC}_0$

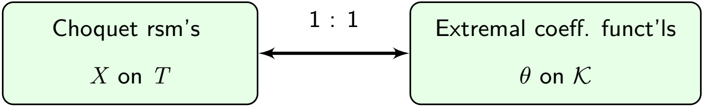
$$\mathbf{E}\left(\int^e f dY\right) = \mathbf{E}\left(\int f dY\right).$$

Corresponding functionals.

$\ell(f) = \mathbf{E}\left(\int^e f dY\right)$

$\theta(K) = \mathbf{E}\left(Y(K)\right)$

New from Old



The set of ecfs θ on $\mathcal{K}$ is closed under

- convex combinations and scaling (trivial)
- operation of Bernstein functions g with $g(0) = 0$ by

$$\theta \mapsto g \circ \theta$$

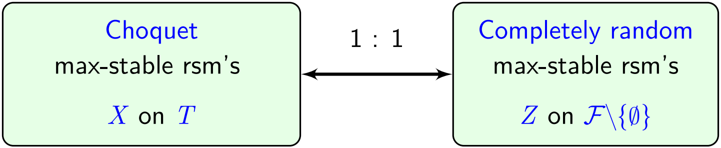
Example:

If θ is an ecf, then also

- $\log(1 + \theta)$ and
- θ^q for $q \in (0, 1]$.

Choquet rsm's as completely random rsm's

There is a one-to-one correspondence:



The correspondence is given by

$$X(K) = Z(\{F : F \cap K \neq \emptyset\}), \qquad K \in \mathcal{K}.$$

Note.

- $\mathcal{F}\backslash\{\emptyset\}$ is a much richer space.
- Z has control measure ν_θ with $\theta(K) = \nu_\theta(\{F : F \cap K \neq \emptyset\})$.
- $\int^e f dX = \int^e f^\vee dZ$, $f \in \mathrm{USC}_0$

Max-stable random sup-measures

Central object.

X = a locally finite max-stable random sup-measure

Properties (almost surely):

- X assumes values in $[0, \infty]$, $X(\emptyset) = 0$,
  X is non-decreasing, but locally finite: $X(K) < \infty$, $K \in \mathcal{K}$
- $K_n \downarrow K$ implies $X(K_n) \downarrow X(K)$
- maxitivity: $X(K_1 \cup K_2) = X(K_1) \vee X(K_2)$
- max-stability: $X^{(1)} \vee X^{(2)} \vee \cdots \vee X^{(n)} \stackrel{\mathcal{D}}{=} n\, X$

Tail dependence functional

$f \in \mathrm{USC}_0(T)$ (usc, bounded, compact support)

Extremal integral:
[Gerritse '96]

$$\int^e f \; dX = \sup_{x>0} x \, X(\{f \geqslant x\}) = \bigvee_{t \in T} f(t)\xi(t)$$

Tail dependence functional:

$$\mathbf{P}\left\{ \int^e f \; dX \leqslant x \right\} = \exp\left(-\frac{\ell(f)}{x} \right)$$

Properties.

$$\textstyle\int^e cf \; dX = c \int^e f \; dX \qquad (\int^e \text{ homogeneus})$$

$$\textstyle\int^e f \vee g \; dX = \int^e f \; dX \vee \int^e g \; dX \qquad (\int^e \text{ max-additive})$$

$$\ell(cf) = c\ell(f) \qquad (\ell \text{ homogeneous})$$

$$\ell(f+g) \leqslant \ell(f) + \ell(g) \qquad (\ell \text{ sub-additive})$$

Duality

The tdf ℓ is an usc sub-linear functional on USC_0.

$\Rightarrow$ By [Fugledge '77]

$$\ell(f) = \sup_{\mu \in \mathbf{M}} \int f d\mu \qquad \text{for} \qquad \mathbf{M} = \left\{ \mu : \int f d\mu \leqslant \ell(f),\ f \in \mathrm{USC}_0 \right\}.$$

TM case.

$$\mathbf{M} = \left\{ \mu : \mu(K) \leqslant \theta(K),\ K \in \mathcal{K} \right\}.$$

Proof based on [Graf '80]

LePage representation of finite Choquet rsm's

$$X(K) \stackrel{\mathcal{D}}{=} \theta(\mathbb{E}) \bigvee_{i \geqslant 1} \Gamma_i^{-1} \mathbb{1}_{\Xi_i \cap K \neq \emptyset}, \qquad K \in \mathcal{K},$$

where

- $\{\Gamma_i, i \geqslant 1\} \sim \mathrm{PPP}(\lambda)$ on $\mathbb{R}_+$

and, independently,

- $\{\Xi_i, i \geqslant 1\} \sim$ i.i.d. random closed sets with capacity functional

$$\mathbf{P}\left\{\Xi \cap K \neq \emptyset\right\} = \frac{\theta(K)}{\theta(\mathbb{E})}.$$

Another limit theorem

Theorem

For independent

- *random sets* $\{\Xi_i^{(n)}, i \geqslant 1\} \sim$ *i.i.d.* $\Xi^{(n)}$ *with* $\mathbf{P}\left\{\Xi^{(n)} \in \cdot\right\} \xrightarrow{vag} \nu$ *(*ν *locally finite),*
- *and marks* $\{\zeta_i, i \geqslant 1\} \sim$ *i.i.d.* ζ *with* $n\mathbf{P}\{\zeta > a_n x\} \to x^{-1}$ *(unit Fréchet domain of attraction),*

the sequence of random sup-measures

$$X_n(K) = a_n^{-1} \bigvee_{i=1}^{nb_n} \zeta_i \mathbb{1}_{\Xi_i^{(n)} \cap K \neq \emptyset}, \qquad K \in \mathcal{K},$$

converges weakly to the Choquet rsm X *with ecf* $\theta(K) = \nu(\mathcal{F}_K)$*.*

- $b_n = 1$: covers weak convergence of $\Xi^{(n)}$ to a random closed set Ξ
- $b_n = n$: covers point process convergence $\sum_{i=1}^{n} \delta_{\Xi_i^{(n)}}$ to a PPP on $\mathcal{F}$ with intensity measure ν