The diameter of an elliptical cloud

Philippe Soulier
Université Paris Ouest Nanterre

Joint work with Yann Demichel et Ana Karina Fermin

Toronto, May 6th 2016

The problem

Let $\mathbb{X}_1, \ldots, \mathbb{X}_n$ be i.i.d. vectors in $\mathbb{R}^d$. The diameter is defined by

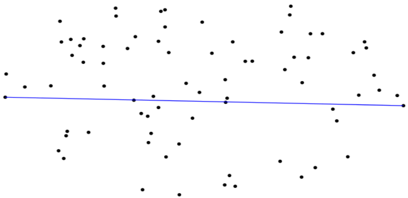
$$D_n(\mathbb{X}) = \max_{1 \le i < j \le n} \|X_i - X_j\| \, ,$$

where $\| \cdot \|$ is any norm.

Goal: asymptotic behaviour of D_n as $n \to \infty$.

This is a non standard extreme value problem since the pairs (X_i, X_j) are dependent. One may expect limit laws which are not extreme value or max-stable distributions.

What can be expected



- If $\max_{1 \le i \le n} \|X_i\|/a_n \to_P 1$, then $D_n = O_P(2a_n)$.
- If the cloud is "more or less isotropic", the origin must be roughly at the center of the cloud and the diameter must be of the order of magnitude of twice the largest norm of the points of the sample. More precisely: then we expect that $D_n/(2a_n)$ converges in probability, possibly to 1, for "more or less isotropic" clouds.

Earlier results on the diameter

- For bounded support: Molchanov and Mayer (not discussed here).
- Max domain of attraction of the Fréchet distribution (next slides).
- Spherical random variables in the domain of attraction of the Gumbel distribution: Asymptotic distribution of the maximum interpoint distance in a sample of random vectors with a spherically symmetric distribution
  Sreenivasa Rao Jammalamadaka and Svante Janson
  Annals of applied probability 2015.

Fréchet case

A random vector X is in the max-domain of attraction of a multivariate Fréchet distribution iff there exists a probability measure σ on the unit $\mathbb{S}^{d-1}$ (relative to the chosen norm) such that

$$n\mathbb{P}\left(\|X\| > a_n x, \frac{X}{\|X\|} \in \cdot\right) \Rightarrow x^{-\alpha}\sigma \ .$$

The point process of exceedences $N_n = \sum_{i=1}^{n} \delta_{\frac{\|X_i\|}{a_n}, \frac{X_i}{\|X_i\|}}$ converges weakly to a Poisson point process $N = \sum_{i=1}^{\infty} \delta_{\Gamma_i^{-1/\alpha}, \Theta_i}$
with mean measure $d(-x^{-\alpha]}) \otimes \sigma$,
$\Gamma_1, \Gamma_2, \ldots$ are the point of a mean 1 PPP on $[0, \infty)$,
$\Theta_1, \Theta_2, \ldots,$ are i.i.d. with law σ on $\mathbb{S}^{d-1}$, independent of the PPP.

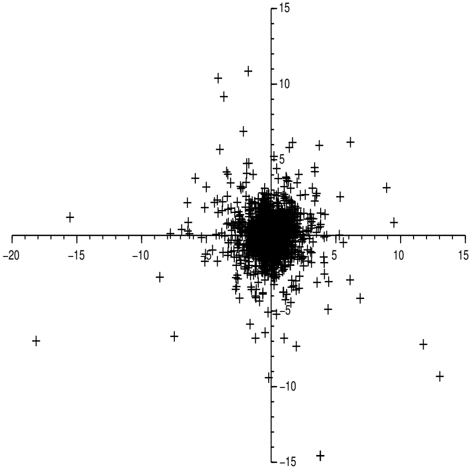
The limiting point process N has a finite number of points outside any ball $B(0, \epsilon)$. This easily yields that

$$\frac{D_n}{a_n} \Rightarrow \max_{i<j} \|\Gamma_i^{-1/\alpha}\Theta_i - \Gamma_j^{-1/\alpha}\Theta_j\| \ .$$

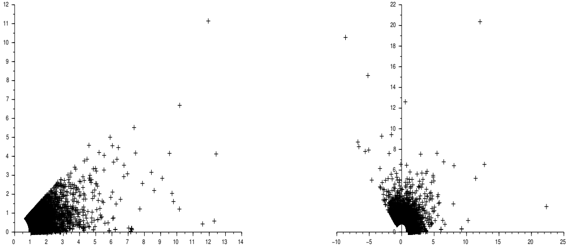
The maximum is achieved over a finite (random) number of pairs of points.

The limiting distribution is in the domain of attraction of the Fréchet distribution and depends on the chosen norm.

Partial results in Henze and Klein (1996), Henze and Lao (2010, unpublished).



10000 spherical Fréchet points. $\alpha = 3$.



Pareto vecteurs supported on cones with angle $\pi/4$ (left) and $2\pi/3$ right.

Gumbel case

A random vector $\mathbb{X}$ is in the multivariate domain of attraction of the Gumbel distribution if there exist sequences a_n, b_n such that

$$\frac{\max_{1\le i\le n}\mathbb{X}_i - b_n}{a_n} \Rightarrow G$$

where G is a multivariate max-stable law with Gumbel marginals.

The point process of exceedences converges weakly

$$\sum_{i=1}^{n} \delta_{\frac{\mathbb{X}_i - b_n}{a_n}} \Rightarrow \sum_{i=1}^{\infty} \delta_{\Gamma_i} \ ,$$

where $\Gamma_i, i \ge 1$ are the points of a PPP with mean measure $-\log G$ on $[-\infty,\infty]^d \setminus \{-\infty\}$.

However, the diameter of the limiting point process is infinite. The same continuous mapping argument cannot be used here.

Euclidean spherical random vectors

A random vector $\mathbb{X}$ is spherically distributed with respect to the euclidean norm on $\mathbb{R}^d$ if it can be expressed as $\mathbb{X} = R\mathbb{W}$, where $\mathbb{W}$ is uniformly distributed on the unit sphere $\mathbb{S}^{d-1}$ and independent of R.

The level lines of the density are spheres.

$\mathbb{X}$ is in the multivariate domain of attraction of the Gumbel law iff R is in the univariate domain of attraction of the Gumbel law. Let $F(x) = \mathbb{P}(R \leq x)$,

$$b_n = F^{\leftarrow}(1 - 1/n) \; , \quad a_n = \psi(b_n) \; ,$$

where ψ is a so-called auxiliary function[1] such that $\psi(x)/x \to 0$ and

$$\lim_{n\to\infty} n\mathbb{P}(R > b_n + a_n x) = \mathrm{e}^{-x} \; .$$

[1] e.g. $\psi = (1 - \tilde{F})/\tilde{F}'$, where $1 - \tilde{F}$ is absolutely continuous and equivalent at ininfity to $1 - F$

Theorem (JJ12)

- If $d \geq 2$,

$$\frac{D_n^{(2)}(\mathbb{X}) - 2b_n}{a_n} + \frac{d-1}{2} \log \frac{a_n}{b_n} - \log\log \frac{a_n}{b_n} - \log c_d \Rightarrow \Lambda \,,$$

where Λ has a standard Gumbel law and $c_d = \frac{(d-1)2^{d-4}\Gamma(d/2)}{\sqrt{\pi}}$.

Theorem (JJ12)

- If $d \geq 2$,

$$\frac{D_n^{(2)}(\mathbb{X}) - 2b_n}{a_n} + \frac{d-1}{2} \log \frac{a_n}{b_n} - \log\log \frac{a_n}{b_n} - \log c_d \Rightarrow \Lambda \ ,$$

  where Λ has a standard Gumbel law and $c_d = \frac{(d-1)2^{d-4}\Gamma(d/2)}{\sqrt{\pi}}$.
- If $d = 1$, then $\mathbb{W} = \pm 1$ with probability $1/2$ and

$$\frac{D_n^{(2)}(\mathbb{X}) - 2b_n}{a_n} \Rightarrow \Lambda_+ + \Lambda_- \ ,$$

  where Λ_- et Λ_+ are indepedent random variable with Gumbel law.

- The proof for $d \geq 2$ uses a criterion for convergence of U statistics to a Poisson distribution and relies heavily on the independence of R and $\mathbb{W}$.
- The uniformlity assumption on $\mathbb{W}$ can be easily relaxed but not the independence of R and $\mathbb{W}$.
- At the end of their paper, J & J ask what happens in the non spherical case, e.g. for a non standard Gaussian distribution.

Elliptical distributions

The simple case of a non standard Gaussian distribution is not covered by the previous result. This is a particular case of an ellipical distribution in $\mathbb{R}^d$ which can be expressed as

$$\mathbb{X} = RA\mathbb{W} ,$$

where as before $\mathbb{W}$ is uniformly distributed on $\mathbb{S}_2^{d-1}$ and independent of R and A is a full rank $d \times d$ matrix.

If R^2 has a χ^2 law with d degrees of freedom, then $\mathbb{X}$ is a Gaussian vector.

The vector $\mathbb{X}$ has a (euclidean) polar representation

$$\mathbb{X} = T\mathbb{U}$$

with $T = \|\mathbb{X}\|_2$ and $\mathbb{U} = \mathbb{X}/\|\mathbb{X}\|_2$ but T and $\mathbb{U}$ are not independent and $\mathbb{U}$ is not uniform.

- In order to study the diameter, we must find where the points with a very large norm are.

Another representation

Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d \geq 0$ be the ordered eigenalues of AA'.

Without loss of generality, we can assume that $A = P\mathrm{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_d})$ where $PP' = I_d$. Set $W = P'\mathbb{W}$ and

$$\mathbb{Y} = P\mathbb{X} = R(\sqrt{\lambda_1}W_1, \ldots, \sqrt{\lambda_d}W_d) \; .$$

Let $\{\mathbb{X}_i, i \geq 1\}$ be i.i.d. with the same distribution as $\mathbb{X}$ and $\mathbb{Y}_i = P\mathbb{X}_i$. Then

$$\|\mathbb{Y}\|_2^2 = \|\mathbb{X}\|_2^2 = R^2 \sum_{i=1} \lambda_i^2 W_i^2 \; ,$$

$$\|\mathbb{Y}_1 - \mathbb{Y}_2\|_2 = \|\mathbb{X}_1 - \mathbb{X}_2\|_2 \; ,$$

$$D_n^{(2)}(\mathbb{X}) = D_n^{(2)}(\mathbb{Y}) \; .$$

Assume that $\lambda_1 = \cdots = \lambda_k > \lambda_{k+1} \geq \cdots \geq \lambda_d$. Then

$$\|\mathbb{Y}\|_2^2 = \lambda_1 R^2 (W_1^2 + \cdots + W_k^2) + R^2 \sum_{i=k+1}^{d} \lambda_i^2 W_i^2 \; .$$

Since $W \in \mathbb{S}^{d-1}$, $\|\mathbb{Y}\|_2$ is largest when $W_{k+1} = \cdots = W_d = 0$.

Thus we can write $\|\mathbb{Y}\|_2 = RV$ with V bounded by λ_1.

Fondamental property of the domain of attraction of the Gumbel law: rapid variation. For all $a > 1$,

$$\lim_{x\to\infty} \frac{\mathbb{P}(R > ax)}{\mathbb{P}(R > x)} = 0 \,.$$

This implies that if R and V are nonnegative independent random variables and V is bounded by $a > 0$, then, conditionally on RV large, V is concentrated close to its maximum:

$$\lim_{x\to\infty} \frac{\mathbb{P}(RV > x,\ V \leq a - \epsilon)}{\mathbb{P}(RV > x)} = 0$$

If V has a density in a neighbourhood of a with a certain smoothness, the rate of convergence and limiting distribution of V given RV is large can be obtained.

Berman (1983), Hashorva (2006,2007,2008),
Fougères and S. (2010), Barbe et Seifert (2013).

Example

Let R be in the Gumbel domain of attraction with auxiliary function ψ and let Θ be uniformly distributed on $[-\pi, \pi]$. What is the asymptotic behaviour of Θ given that $R \cos \Theta > x$ as $x \to \infty$? Set $\phi(x) = \sqrt{\psi(x)/x}$. Then

$$
\begin{aligned}
\mathbb{P}(0 \le \Theta \le \phi(x)u, R\cos\Theta > x) &= \int_0^{\phi(x)u} \mathbb{P}(R > x/\cos v)\mathrm{d}v \\
&= \phi(x) \int_0^u \mathbb{P}(R > x/\cos(\phi(x)v))\,\mathrm{d}v \\
&\sim \phi(x) \int_0^u \mathbb{P}(R > x/(1 - \phi^2(x)v^2/2))\,\mathrm{d}v \\
&\sim \phi(x) \int_0^u \mathbb{P}(R > x + \psi(x)v^2/2)\,\mathrm{d}v \\
&\sim \phi(x)\mathbb{P}(R > x) \int_0^u \mathrm{e}^{-t^2/2}\,\mathrm{d}v\ .
\end{aligned}
$$

Theorem

As $x \to \infty$, conditionally on $\|\mathbb{Y}\| > x$,

$$\left(\frac{\|\mathbb{Y}\| - x}{\psi_A(x)}, W_1, \dots, W_k, \frac{W_{k+1}}{\phi_A(x)}, \dots, \frac{W_d}{\phi_A(x)} \right) \Rightarrow \left(E, \mathbb{W}^{(k)}, \tau_{k+1} G_{k+1}, \dots, \tau_d G_d \right), \quad (*)$$

where all components are independent and

- E has a standard exponential distribution,
- $\mathbb{W}^{(k)}$ is uniformly distributed on $\mathcal{S}_2^{k-1}$,
- $G_{k+1}, \dots, G_d$ are i.i.d. N(0,1),
- $\psi_A(x) = \psi(x/\sqrt{\lambda_1})$, $\phi_A(x) = \sqrt{\psi_A(x)/x} \to 0$,
- $\tau_i = \sqrt{\frac{\lambda_1}{\lambda_1 - \lambda_i}}$, $i = k+1, \cdots, d$.

The vectors with large norm concentrate on the eigenspace corresponding to the largest eigenvalue.

The rate of convergence $\phi_A(x)$ and limiting distribution of $W_i/\phi_A(x)$ for $i > k$ is determined by the geometry of the level lines.

Consequences for the diameter

Two cases:

- If $k = 1$, the large vectors are concentrated around the principal axis. One can expect a one dimensional type result.
- If $k > 1$, the vectors with large norm concentrate in a subspace of dimension k and behave like euclidean spherical vectors in this subspace. On can expect a result close to the spherical case.

From the norm to the diameter: $k = 1$

Consider the polar representation $\mathbb{Y}_i = \|\mathbb{Y}_i\|_2 \Theta_i$, with $\|\Theta_i\|_2 = 1$, non uniform and not independent of $\|\mathbb{Y}_i\|_2$.

Conditionally on $\|\mathbb{Y}_i\|_2$ being large, $\mathbb{Y}_i$ is close to the line $x_2 = \cdots = x_n = 0$. We build two point processes, one on each half-plane. Formally, set $c_n = \phi_A(b_n) = \sqrt{a_n/b_n}$ and

$$P_{n,i} = \left(\frac{\|\mathbb{Y}_i\|_2 - b_n}{a_n}, \Theta_{i,1}, \frac{\Theta_{i,2}}{c_n}, \dots, \frac{\Theta_{i,d}}{c_n}\right) \ ,$$

$$N_n^+ = \sum_{i=1}^n \delta_{P_{n,i}} \mathbb{1}_{\{\Theta_{i,1} > 0\}} \ ,$$

$$N_n^- = \sum_{i=1}^n \delta_{P_{n,i}} \mathbb{1}_{\{\Theta_{i,1} > 0\}} \ .$$

Point process convergence

The weak convergence $(*)$ yields

$$(N^+, N^-) \Rightarrow \left(\sum_{i=1}^{\infty} \delta^+_{P_i}, \sum_{i=1}^{\infty} \delta^-_{P_i} \right)$$

with

- $P_i^{\pm} = (\Gamma_i^{\pm}, \pm 1, \tau_2 G_{i,2}^{\pm}, \ldots, \tau_d G_{i,d}^{\pm})$,
- $\sum_{i=1}^{\infty} \delta_{\Gamma_i^{\pm}}$ are two independent PPP on $(-\infty, \infty]$ with mean measure $\frac{1}{2}\mathrm{e}^{-x}\mathrm{d}x$.

The points can be ordered $\Gamma_1^{\pm} > \Gamma_2^{\pm} > \ldots$ since there is a finite number of points in each interval $[a, \infty]$ with $a > -\infty$.

Then

$$\mathbb{P}(\Gamma_1^{\pm} \leq x) = \mathrm{e}^{-\mathrm{e}^{-x-\log 2}} .$$

The support function

Let $\mathbb{X}_1, \ldots, \mathbb{X}_n$ be random points in $\mathbb{R}^d$. The support function S_n of the convex hull of the sample is defined on $\mathcal{S}^{d-1}$ by

$$S_n(\mathbb{W}) = \max_{1 \le i \le n} \langle \mathbb{X}_i, \mathbb{W} \rangle = \max_{1 \le i \le K} \langle \tilde{\mathbb{X}}_i, \mathbb{W} \rangle \ , \ \mathbb{W} \in \mathcal{S}^{d-1} \ ,$$

where $\tilde{\mathbb{X}}_i, i = 1, \ldots, K$ are the vertices of the convex hull (K is random). The diameter can be expressed in terms of the support function:

$$D_n = \max_{\mathbb{W} \in \mathcal{S}_+^{d-1}} \{ S_n(W) + S_n(-W) \} \ .$$

For simplicity, we proceed in the case $d = 2$. Write $\mathbb{X}_i = (X_i, Y_i)$ and using the polar representation we have

$$S_n(\theta) = \max_{1 \le i \le n} (X_i \cos\theta + Y_i \sin\theta) \ ,$$

$$D_n = \max_{0 \le \theta \le \pi} \{ S_n(\theta) + S_n(\theta + \pi) \} \ .$$

Convergence of the support function

Spherical case (Eddy and Gale, 1981)

In the case $d = 2$, write $S_n(\theta) = S_n(\cos\theta, \sin\theta)$. Then

$$\{S_n(c_n t) - b_n\}/a_n \Rightarrow M(t) \;, \tag{1}$$

locally uniformly, where $a_n = \psi(b_n)$, $c_n = \sqrt{a_n/b_n}$ and M is a max-stable process

$$M(t) = \sup_{i \geq 1}\{\Gamma_i + Z_i t - t^2/2\} \;,$$

where $\{\Gamma_i, i \geq 1\}$ are the decreasing points of a PPP on $(-\infty, \infty]$ with mean measure $\mathrm{e}^{-x}\mathrm{d}x$ and $\{Z_i, i \geq 1\}$ are i.i.d. standard Gaussian random variables, independent of the PPP. The process M is stationary and ergodic and $M(0)$ has a standard Gumbel distribution.

In the spherical case, this can't be used to study the diameter.

Elliptical case, $d = 2$

In polar coordinates, the previous point process convergence can be rewritten as

$$N_n^+ = \sum_{i=1}^{n} \delta_{R_{n,i},\Theta_{n,i}} \mathbb{1}_{\{|\Theta_i - \pi/4| \le \pi/2\}} \Rightarrow N^+ = \sum_{i=1}^{\infty} \delta_{\Gamma_i^+, Z_i^+} ,$$

$$N_n^- = \sum_{i=1}^{n} \delta_{R_{n,i},\Theta_{n,i}} \mathbb{1}_{\{|\Theta_i - 5\pi/4| \le \pi/2\}} \Rightarrow N^- = \sum_{i=1}^{\infty} \delta_{\Gamma_i^-, Z_i^-} ,$$

with $R_{n,i} = \frac{R_i - b_n}{a_n}$, $\Theta_{n,i} = \frac{\Theta_i - \pi/4}{c_n}$, $\{\Gamma_i^+, i \ge 1\}$, $\{\Gamma_i^-, i \ge 1\}$ are the points of two independent PPP on $(-\infty, \infty]$ with mean measure $\frac{1}{2}\mathrm{e}^{-x}$ and $\{Z_i^+, i \ge 1\}$ $\{Z_i^+, i \ge 1\}$ are two independent sequences of i.i.d. Gaussian random variables with mean zero and variance $(1 - \rho)/(2\rho)$.

Convergence of the support function (2)

$$\left\{\left(\frac{S_n(\pi/4 + c_n t) - b_n}{a_n}, \frac{S_n(5\pi/4 + c_n t) - b_n}{a_n}\right), t \in \left[-\frac{\pi}{2c_n}, \frac{\pi}{2c_n}\right]\right\}$$
$$\Rightarrow \{(M_+(t), M_-(t)), t \in \mathbb{R}\} \ ,$$

where the convergence is locally uniform and M_+ and M_- are two independent max-stable processes with the same distribution, which can be expressed as

$$M_\pm(t) = \vee_{i=1}^{\infty}\{\Gamma_i^\pm - \frac{1}{2}(t - Z_i^\pm)^2\} \tag{2}$$

Proof.

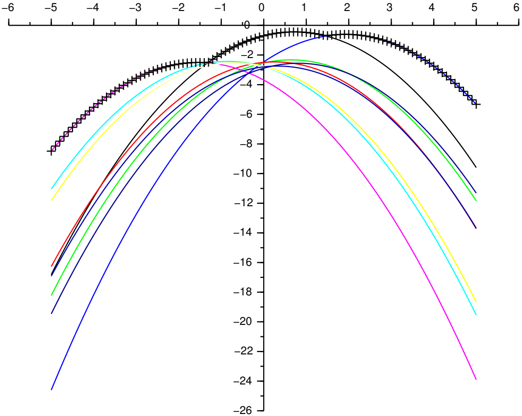
For $t \in [-\pi/(2c_n), \pi/2(c_n)]$,

$$
\begin{aligned}
&a_n^{-1}\{S_n(\pi/4 + c_n t) - b_n\} \\
&\quad = a_n^{-1} \max_{i=1,\dots,n} \{(b_n + a_n R_{n,i}) \cos(c_n(t - \Theta_{n,i})) - b_n\} \\
&\quad = \max_{i=1,\dots,n} \{R_{n,i} \cos(c_n(t - \Theta_{n,i})) - c_n^{-2}\{1 - \cos(c_n(t - \Theta_{n,i}))\} \\
&\quad \to \max_{i \geq 1} \{\Gamma_i^+ - \frac{1}{2}(t - Z_i^+)^2\} .
\end{aligned}
$$

The convergence is locally uniform. $\square$

The limiting processes M_+ and M_- are non stationary. If $\rho \in (0, 1)$,

$$
\mathbb{P}\text{--} \lim_{t \to \infty} t^{-2} M_+(t) = -\rho/(1 + \rho)
$$



Le processus $M_{\pm}$.

Recall that the diamter is given by

$$D_n = \max_{1 \leq i < j \leq n} \|X_i - X_j\| = \max_{0 \leq \theta \leq \pi} \{S_n(\theta) + S_n(\theta + \pi)\} \ .$$

Recall that the diamter is given by

$$D_n = \max_{1 \le i < j \le n} \|X_i - X_j\| = \max_{0 \le \theta \le \pi} \{S_n(\theta) + S_n(\theta + \pi)\} .$$

Theorem

$$\frac{D_n - 2b_n}{a_n} \Rightarrow \max_{t \in \mathbb{R}} \{M_+(t) + M_-(t)\}$$
$$= \max_{i,j \ge 1} \{\Gamma_i^+ + \Gamma_j^- - \frac{1-\rho}{8\rho}(G_i^+ - G_j^-)^2\} ,$$

where $G_i^{\pm}$ are i.i.d. standard Gaussian random variables

Proof.

$$
\begin{aligned}
&\sup_{t\in\mathbb{R}}\{M_+(t)+M_-(t)\} \\
&\quad = \sup_{t\in\mathbb{R}}\left\{\sup_{i\geq 1}\Gamma_i^+ - \frac{1}{2}(t-Z_i^+)^2 + \sup_{j\geq 1}\Gamma_j^+ - \frac{1}{2}(t-Z_j^+)^2\right\} \\
&\quad = \sup_{i\geq 1}\sup_{j\geq 1}\sup_{t\in\mathbb{R}}\left\{\Gamma_i^+ - \frac{1}{2}(t-Z_i^+)^2 + \Gamma_j^+ - \frac{1}{2}(t-Z_j^+)^2\right\} \\
&\quad = \sup_{i\geq 1}\sup_{j\geq 1}\left\{\Gamma_i^+ + \Gamma_j^+ - \frac{1}{2}(Z_i^+ - Z_j^-)^2\right\}.
\end{aligned}
$$

The last equality holds since $\inf_{t\in\mathbb{R}}\{(t-Z_i^+)^2+(t-Z_j^+)^2\} = (Z_i^+ - Z_j^-)^2/4$. The convergence of S_n to $M^+ + M^-$ is only locally uniform, but this is sufficient because M_+ and M_- are tangent to downwards parabolae. $\square$

$d > 2$, $k = 1$

Recall that $\lambda_1 > \lambda_2 \geq \cdots \geq \lambda_d$ are the eigenvalues of the covariance matrix.

Theorem

$$\frac{D_n - 2b_n}{a_n} \to \max_{i,j \geq 1} \left(\Gamma_i^+ + \Gamma_j^- - \frac{1}{4} \sum_{q=2}^{d} \frac{\lambda_q}{\lambda_1 - \lambda_q} (Z_{i,q}^+ - Z_{j,q}^-)^2 \right) .$$

the max being almost surely finite and achieved.

There is a discontinuity when $\lambda_1 \to \lambda_2$.

Case $k > 1$

We recover the result of JJ12 with adapted constants.

$$\frac{D_n^{(2)}(\mathbb{X}) - 2b_n}{a_n} + d_n \Rightarrow \Lambda \,,$$

where Λ has a Gumbel law,

$$d_n = \frac{k-1}{2} \log \frac{a_n}{b_n} - \log\log \frac{a_n}{b_n} - \log C_k \,,$$

$$C_k = \frac{(2d-k-1)2^{k-4}\Gamma(k/2)}{\sqrt{\pi}} \left(\prod_{q=k+1}^{d} \frac{\lambda_1}{\lambda_1 - \lambda_q} \right)^{-1/2} .$$

Proof adapted from JJ12: classical criterion for Poisson convergence of U-statistics.

Generalization

Consider a random vector $\mathbb{X}$ which can be expressed as

$$\mathbb{X} = Rg(\mathbb{W})$$

where R is a positive random variable, $\mathbb{W}$ is uniformly distributed on $\mathcal{S}^{d-1}$, and $g : \mathcal{S}^{d-1} \to \mathbb{R}^d$ is a continuous function.

Where is $\mathbb{X}$ when $\|\mathbb{X}\|$ is large?

Depends on the maxima of $h(\mathbb{W}) = \|g(\mathbb{W}\|$:

- achieved at isolated points: concentration at a rate depending on the shape of the level lines of the density;
- achieved on a submanifold: concentration on the submanifold.

Hashorva, Korshunov and Piterbarg (2015) give expansions of $\|\mathbb{X}\|$ but did not consider the location of the large vectors.

Related to "implicit extremes" of Scheffler and Stoev (2014).

The ℓ^q $(q \geq 1)$ norm of euclidean spherical distributions

Let $\mathbb{X} = R\mathbb{W}$ still be a euclidean spherical vector and consider $\|\mathbb{X}\|_q$ and

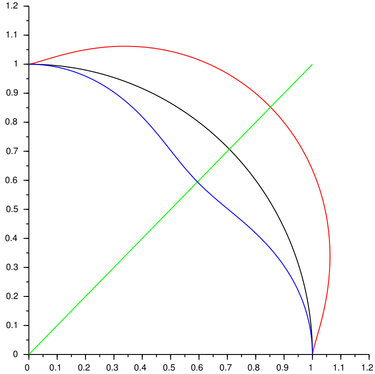
$$D_n^{(q)}(\mathbb{X}) = \max_{1 \leq i < j \leq n} \|\mathbb{X}_i - \mathbb{X}_j\|_q \ .$$

For $d \geq 2$ and $q \geq 1$, $q \neq 2$, the maximum of the l^q norm on the euclidean sphere is achived at isolated points.

- $q \in [1, 2)$: $\max_{w \in \mathcal{S}^{d-1}} \|w\|_q = d^{1/q - 1/2}$, achieved on the 2^d diagonal points $(\pm d^{-1/2}, \ldots, \pm d^{-1/2})$.
- $q \in (2, \infty)$: $\max_{w \in \mathcal{S}^{d-1}} \|w\|_q = 1$, achieved on the $2d$ intersections with the axes.

The diameter is achieved close to diametrically opposed such points.

The ℓ^q norm of ℓ^2 spherical distributions



$q = 1.3$, $q = 4$.

$1 \le q < 2$

$$\frac{D_n^{(q)}(\mathbb{X}) - 2a_n}{b_n} \Rightarrow$$

$$\max_{1 \le j \le 2^{d-1}} \max_{i,i' \ge 1} \left\{ \Gamma_{i,j}^{+} + \Gamma_{i',j}^{-} - \frac{q-1}{4} \sum_{\ell=1}^{d} (G_{i,j,\ell}^{+} + G_{i',j,\ell}^{-})^2 \right\} ,$$

where $\Gamma_{i,j}^{\pm}$, $i \ge 1$ $j = 1, \ldots, 2^{d-1}$ are points of i.i.d. PPP on $(-\infty, \infty]$ with mean measure $2^{-d} \mathrm{e}^{-x} \, \mathrm{d}x$ and $\mathbb{G}_{i,j}^{\pm} = (G_{i,j,1}^{\pm}, \ldots, G_{i,j,d}^{\pm})$, $i \ge 1$, $j = 1, \ldots, 2^{d-1}$ are i.i.d. Gaussian vectors with covariance matrix

$$\frac{1}{d(2-q)} \begin{pmatrix} d-1 & -1 & \ldots & -1 \\ -1 & d-1 & \ldots & -1 \\ \vdots & & & \vdots \\ -1 & \ldots & -1 & d-1 \end{pmatrix} .$$

$q \in (2, \infty]$

$$\frac{D_n^{(q)}(\mathbb{X}) - 2a_n}{b_n} \Rightarrow \max_{1 \le i \le d} \left(\Gamma_i^{+} + \Gamma_i^{-}\right) ,$$

where $\Gamma_i^{\pm} - \log(2d)$, $1 \le i \le d$ are i.i.d. Gumbel random variables.